Today (12/6/05)

- Today
  - Another way to find feedback gain: linear quadratic regulator (LQR)
  - Application to observer design: Kalman Filter
  - Combining LQR and Kalman Filter: Linear Quadratic Gaussian (LQG) controller

Quadratic Optimization

Choosing desired closed poles could be a tedious trial-and-error / hit-and-miss process. Another way to find the feedback gain is to pose the following optimization problem:

$$\min_u \underbrace{\int_0^\infty (\underbrace{x^T Q x}_{\text{State Penalty}} + \underbrace{u^T R u}_{\text{Control Penalty}}) dt}_{\text{Cost Function}}$$

This is called the linear quadratic regulator problem, or LQR.

Quadratic Optimization

Solution to the optimization problem:

$$u = -Fx$$

$$F = R^{-1}B^{T}P$$

where P solves a matrix quadratic equation called the Riccati Equation.

$$A^{T}P + PA + Q - PB^{T}R^{-1}BP = 0$$

P is a symmetric positive definite matrix (i.e., $x^{T}Px$ is positive for all nonzero x).

Optimal cost: $x_0^{T}Px_0$ for the initial state x_0

MATLAB Command

```
F=lqr(A,B,Q,R); % feedback gain

[F,P,E]=lqr(A,B,Q,R);

% P = solution of Riccati Equation

% E = closed loop eigenvalues (i.e.,

%     eig(A-B*F) )
```

Single Link Example

Consider the single link pendulum:

$$\min_u \int_0^\infty \Big(\underbrace{q_1 x_1^{\,2}(t)}_{\text{Weighted position error}} + \underbrace{q_2 x_2^{\,2}(t)}_{\text{Weighted velocity error}} + \underbrace{r u^2(t)}_{\text{Weighted control effort}} \Big) dt$$

Tuning parameters are now the relative weights, (q_1,q_2,r), instead of the closed loop poles.

Large q_2 ➔ slower position response

Small r ➔ faster overall response

```
Q=diag([1,.1]);R=.01;

F=lqr(A,B,Q,R); % feedback gain
```

Robustness

Property of LQR:

Guaranteed (-60^{o},60^{o}) phase margin and (-6dB,inf dB) gain margin (allowable gain variation from –1/2 to infinity).

So we get the robustness for free! We can concentrate on choosing Q and R to get the time domain performance.

Kalman Filter

We can use the same approach to choose the observer gain. Consider a state space system corrupted by noise:

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + Du + v$$

w = state noise, v = output noise

Both are assumed to be zero mean, white, Gaussian noise with covariance W and V.

$w(t)$ and $v(t)$ both are zero mean Gaussian random variables.

$$E[w(t)w^T(\tau)] = \delta(t-\tau)W,\ E[v(t)v^T(\tau)] = \delta(t-\tau)V$$

Kalman Filter

We can use the same approach to choose the observer gain.

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} + Du - y)$$

$$L = \Sigma C^T V^{-1}$$

where Σ solves the dual Riccati Equation:

$$A\Sigma + \Sigma A^T + W - \Sigma C^T V^{-1} C \Sigma = 0$$

LQG Control

Combining LQR and Kalman Filter: Linear Quadratic Gaussian (LQG) controller

$$\dot{\hat{x}} = (A - LC - (B - LD)F)\hat{x} + Ly$$

where F is the optimal LQR gain and

L is the Kalman gain

However, there is no longer guaranteed stability margin as in LQR.

MATLAB Command

```
F=lqr(A,B,Q,R); % optimal lqr gain

[F,P,E]=lqr(A,B,Q,R);

L=lqr(A’,C’,W,V)’; % Kalman gain

% other related commands

% kalman, lqg
```