Final Exam

- Root Locus
- Bode plots
- Nyquist plots
- Gain / phase margin
- 2nd order system response

  ↔ / Reale time / settling time / overshoot

- Linearization
- State space design
- Control design

  range of gain for stability ⊲ – Routh Criterion

- Different representations of an LTI system
  - input/output differential equation

    $$\dddot{x} + 3\ddot{x} - 2\dot{x} + 3x = 3\dot{u} - u$$

  - impulse response $\xrightarrow{\delta} \boxed{} \xrightarrow{h}$
  - transfer function $\mathcal{L}[h]$
  - state space
- Final Value Theorem

Representation of LTI Systems

- input/output differential equation

$$\dddot{y}+2\ddot{y}+3\dot{y}-4y = 3\ddot{u}-2\dot{u}+u$$

(or $\ddot{z}+2\dot{z}-3z = 4\dot{u}+u$
$\ddot{y} = 4\ddot{z}-2z+u$)

- Transfer function

$$\frac{Y(s)}{U(s)} = \left.\frac{3s^2-2s-1}{s^3+2s^2+3s-4}\right\} H(s)$$

- Impulse response

$$h(t) = \mathcal{L}^{-1}[H(s)] \qquad \mathcal{L}^{-1}\left[\frac{s+2}{s(s+1)}\right]$$

partial fraction expansion

$$= \frac{2}{s} - \frac{1}{(s+1)} \longleftrightarrow 2\,\mathbb{1}(t) - e^{-t}\,\mathbb{1}(t)$$

- state space

$$\dddot{z}+2\ddot{z}+3\dot{z}-4z = u$$

$$y = 3\ddot{z}-2\dot{z}+z$$

$x_1 = z$
$x_2 = \dot{z}$
$x_3 = \ddot{z}$

$\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\dot{x}_3 = 4x_1-3x_2-2x_3+u$

$y = x_1-2x_2+3x_3$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} x$$

$$C(sI-A)^{-1}B + D = H(s)$$

poles of $H(s)$ = eig(A) = roots of $\underline{\det(sI-A)}$

characteristic polynomial
(pole polynomial)

Final Value Theorem

If $y(t) \longrightarrow y_{ss}$ as $t \to \infty$, then

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,\bar{Y}(s)$$

e.g. Step response of $\boxed{\dfrac{s+2}{(s+4)(s+1)}}$ stable x'fer fn

steady state value of step response $= \lim_{s\to 0} \cancel{s} \cdot \dfrac{s+2}{(s+4)(s+1)} \cdot \dfrac{1}{\cancel{s}}$ step

$= \frac{1}{2}$

What about ramp response?

$$\frac{s+2}{(s+4)(s+1)} \boxed{\frac{1}{s^2}} \leftarrow \underline{\underline{\text{ramp}}}$$

does not reach a steady state

However,

$$\lim_{s \to 0} \cancel{s} \frac{\overbrace{\cancel{s}(s+2)}^{x' \text{ for } f'n}}{(s+4)(s+1)} \frac{1}{\cancel{s^2}} = \underline{\underline{\frac{1}{2}}}$$

<u>step response</u>:

$$\lim_{s \to 0} s \frac{\cancel{s}(s+2)}{(s+4)(s+1)} \frac{1}{\cancel{s}} = 0$$

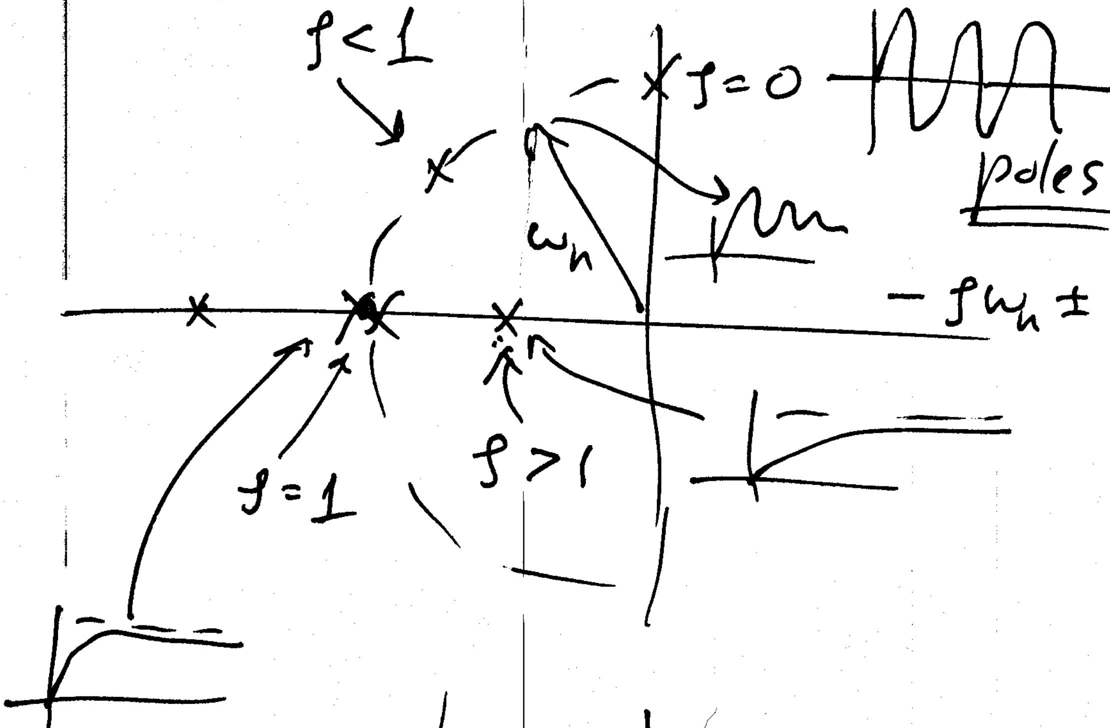
2nd Order systems with no zeros

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$$

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

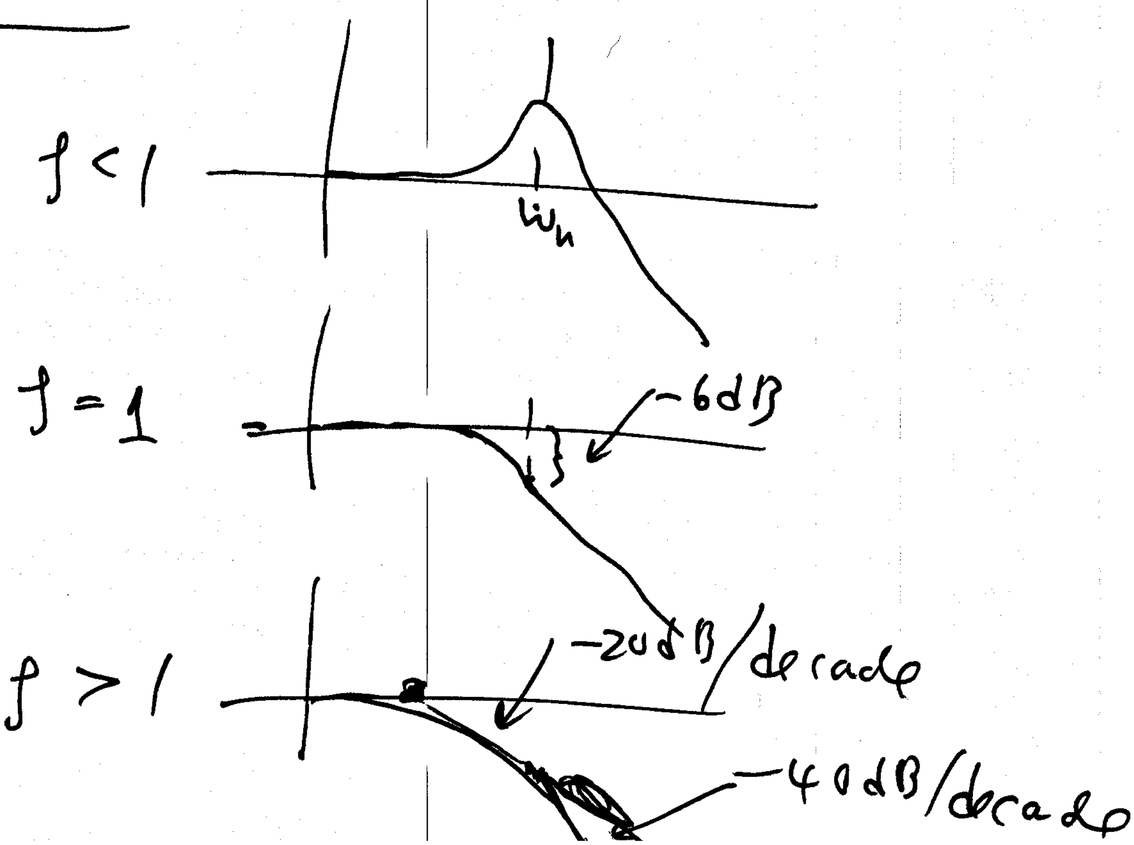
ζ = damping ratio

ω_n = undamped natural frequency

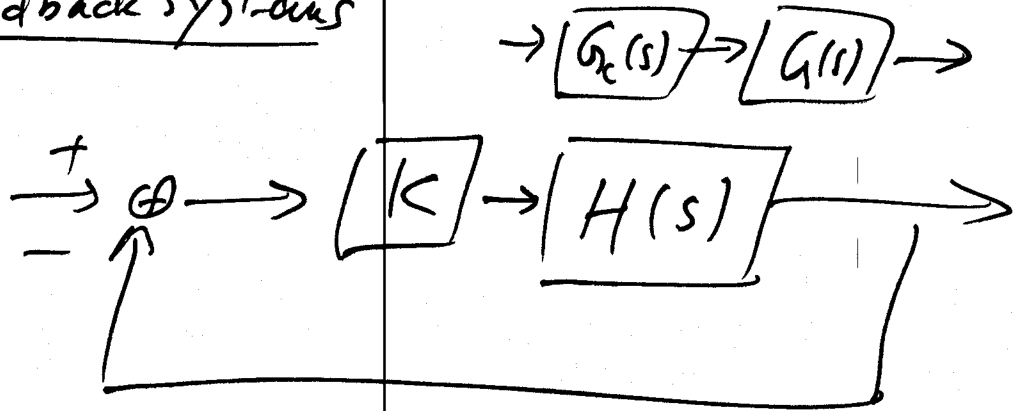


poles:

$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



Feedback Systems



Find the range of K such that the closed loop system is stable.

- Routh criterion
- Bode plot
- Nyquist plot

Example: $H(s) = \dfrac{-s+2}{s(s+1)(s+3)} = \dfrac{-(s-2)}{s(s+1)(s+3)}$

Closed loop characteristic polynomial

zero poly. of $(1+H(s)) = s(s+1)(s+3) + K(-s+2)$

$$= s^3 + 4s^2 + 3s - Ks + 2K$$

$$\begin{array}{c|ccc} s^3 & 1 & 3-K & 0 \\ s^2 & 4 & 2K & 0 \\ s^1 & -\frac{(2K-4(3-K))}{4} & 0 & \\ s^0 & 2K & 0 & \end{array}$$

$K > 0$

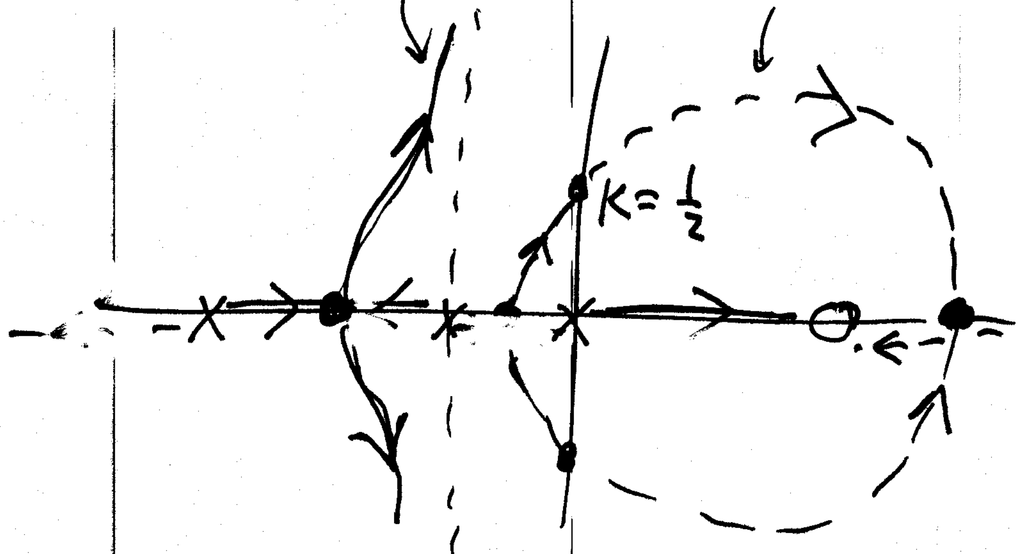
$12 - 6K > 0$

or $K < \frac{1}{2}$

$\boxed{0 < K < \frac{1}{2}}$

8

root locus



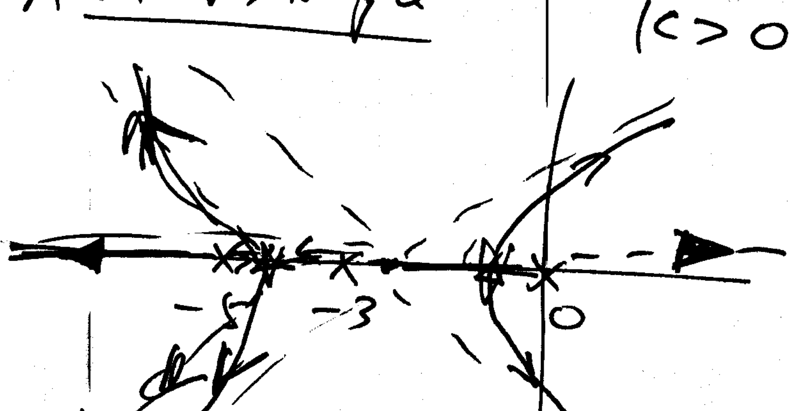
- asymptotes
- intersection of asymptotes
- break pt

$$H(s) = \frac{-(s-2)}{s(s+1)(s+3)}$$

$$\frac{\sum P_i - \sum Z_i}{n-m} = \frac{-4+2}{2} = -1$$

$$\frac{dH()}{ds} = 0 \Rightarrow \frac{s^3+4s^2+s-(s-2)(3s^2+4s+1)}{(s(s+1)(s+3))^2} = 0$$

Another example



$$H(s) = \frac{1}{s(s+3)(s+5)}$$

intersection of asymptote with real axis

$$\frac{\sum P_i - \sum z_i}{n-m} = -\frac{8}{3}$$

$$\frac{dH(s)}{ds} = 0$$

$\Rightarrow \quad 3s^2+16s+15=0$

$(-4.11, -1.21)$

To solve for ω, use Nyquist:

Solve for ω such that $\underline{\operatorname{Im} H(j\omega) = 0}$

$$H(j\omega) = \frac{-(j\omega - 2)}{j\omega(j\omega+1)(j\omega+3)}$$

$$= \frac{-j\omega+2}{j\omega(3-\omega^2+4j\omega)} = \frac{-j\omega+2}{-4\omega^2+j\omega(3-\omega^2)}$$

$$= \frac{\boxed{-8\omega^2 - \omega^2(3-\omega^2)} + \boxed{4j\omega\cdot\omega^2 - 2j\omega(3-\omega^2)}}{\boxed{16\omega^4 + \omega^2(3-\omega^2)^2}}$$

$\operatorname{Im} H(j\omega) = 0 \iff 4\omega^2 - 2(3-\omega^2) = 0$

$6\omega^2 = 6 \iff \underline{\omega = \pm 1}$

at $\omega = 1$

$$\operatorname{Re} H(j) = \frac{-8-2}{16+4} = \underline{\underline{-\frac{1}{2}}}$$

State Space design

$$\frac{-s+2}{s(s+1)(s+3)} \longleftrightarrow \text{State space form}$$

$$(s^3+4s^2+3s)z = u$$

$$y = -\dot{z} + 2z$$

$$x_1 = z$$
$$x_2 = \dot{z}$$
$$x_3 = \ddot{z}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

use full state feedback to place poles at $(-1,-2,-3)$.

$$u = -[f_1 \quad f_2 \quad f_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -f_1 & -3-f_2 & -4-f_3 \end{bmatrix}}_{A_{cl}} x$$

$$\mathrm{eig}(A_{cl}) = (-1,-2,-3)$$

$$\det(sI - A_{cl}) = (s+1)(s+2)(s+3)$$

$$s^3 + (4+f_3)s^2 + (3+f_2)s + f_1 = s^3 + 4s^2 + 11s + 6$$

$$f_1 = 6, \quad f_2 = 8, \quad f_3 = 0$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u \quad [f_1 \; f_2 \; f_3] x$$

$$y = [1 \; 1 \; 1] x$$

$$\longleftrightarrow$$

$$[1 \; 1 \; 1] \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$sI - A_{cl} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ f_1 & 3+f_2 & s+(4+f_3) \end{bmatrix}$$

$$\det(sI - A_{cl}) = \left(s^2 + (4+f_3)s + (3+f_2)\right)s + f_1$$

$$H(s) = \frac{(s+2)10}{(s+4)(s+1)}$$

$$H(j\omega) = \frac{j\omega + 2}{4-\omega^2+5j\omega} = \frac{2(4-\omega^2)+5\omega^2 - 10j\omega + (4-\omega^2)j\omega}{(4-\omega^2)^2+25\omega^2}$$

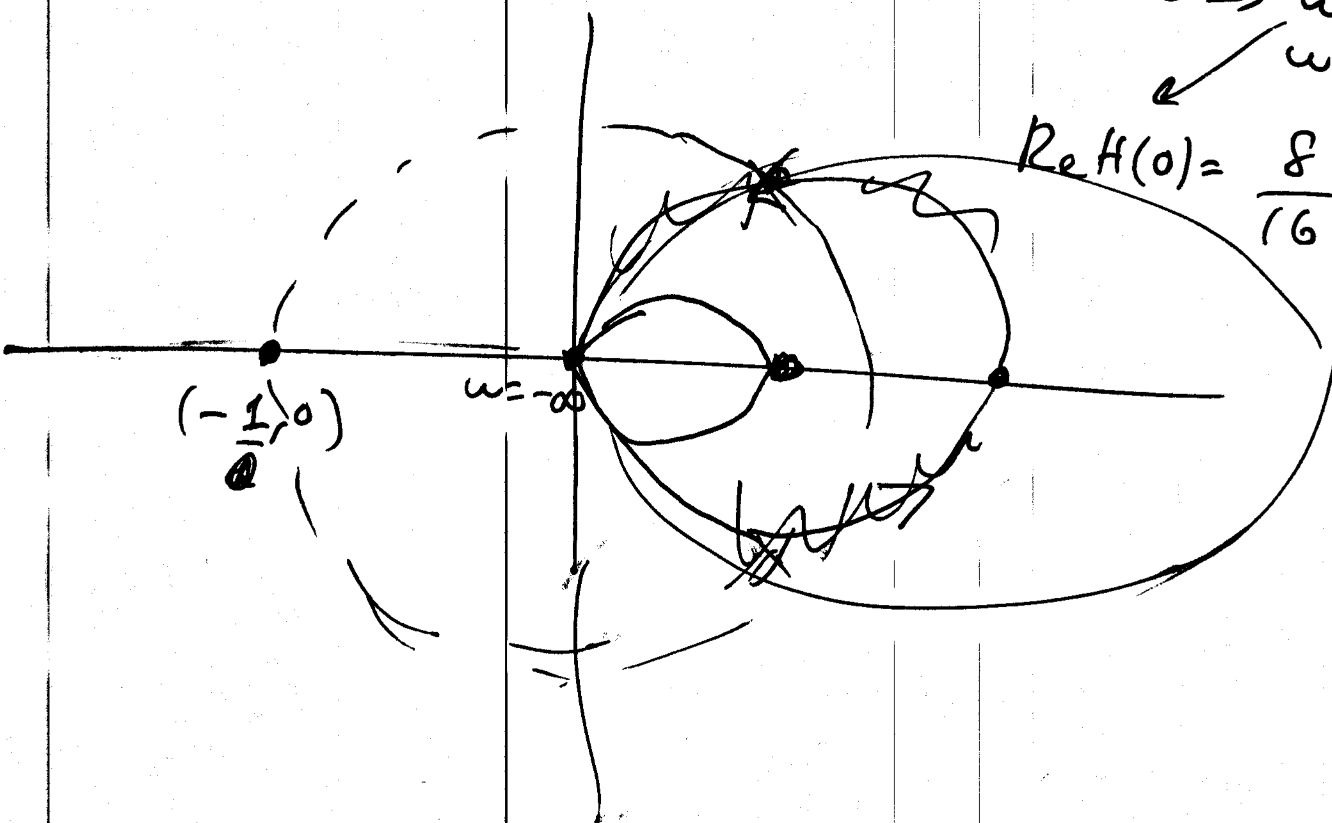
$$= \frac{8+3\omega^2 + j\omega(-6-\omega^2)}{(4-\omega^2)^2+25\omega^2}$$

$\operatorname{Im} H(j\omega) = 0$

$\Leftrightarrow \omega = 0$

$\omega = \pm\infty$

$\operatorname{Re} H(0) = \frac{8}{16} = \frac{1}{2}$



$$H(s) = \frac{5}{(s+4)(s+10)}$$

$$H(j\omega) = \frac{5}{(j\omega+4)(j\omega+10)} = \frac{5}{40-\omega^2+14j\omega}$$

$$= \frac{5\left((40-\omega^2)-14j\omega\right)}{(40-\omega^2)^2+(14\omega)^2}$$

$\mathrm{Re}\, H(j\omega) = 0 \iff \omega = \pm\sqrt{40}$

$\mathrm{Im}\, H(j\omega) = 0 \iff \omega = 0$

$\mathrm{Im} = \frac{\mp 14\sqrt{40}}{(\quad - \quad)}$

$\mathrm{Re} = \frac{200}{1600} = \frac{1}{8}$