Today (11/11/05)

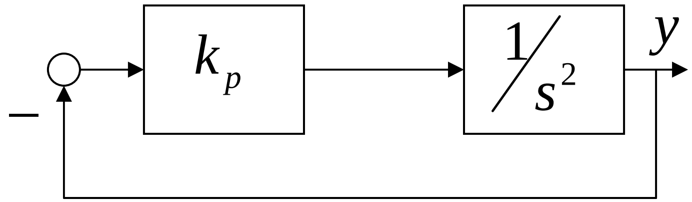
- Get started with project: try the open loop response, see how PID controllers would work.
- Root locus review
- Model of a two-link planar arm
- State space description of an LTI system

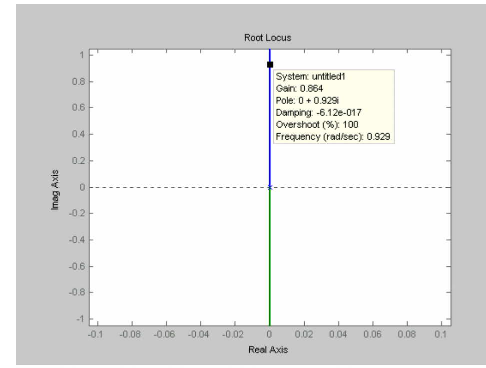
Root Locus

- **Plot the closed loop poles as a function of a *single* gain.**

$$s^2 + k_p = 0$$

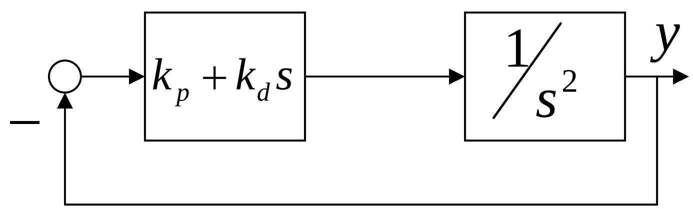
$$1 + L(s) = 1 + \frac{k_p}{s^2}$$





- **What if there are more than one gains?**

$$s^2 + k_d s + k_p = 0$$



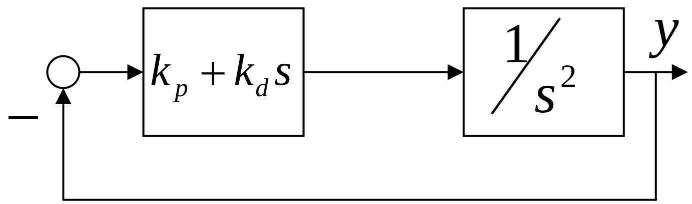
***L(s)* = loop gain, *1+L(s)*= return difference**

Closed loop pole: zeros of *1+L(s)*

Open loop poles: poles of *L(s)* and *1+L(s)*

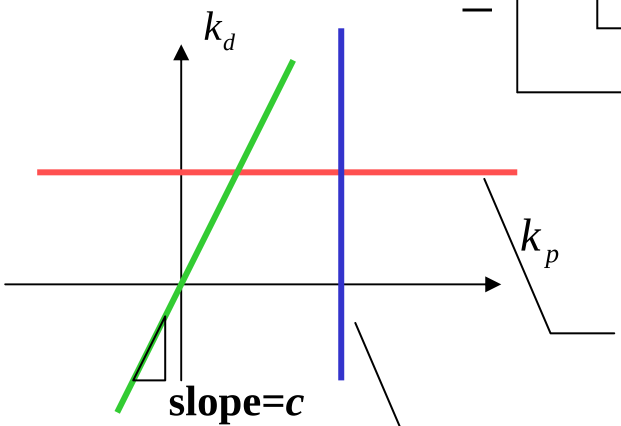
Multiple Gains

$$s^2 + k_d s + k_p = 0$$



$$1 + L(s) = 1 + \frac{k_p + k_d s}{s^2}$$

$$L(s) = \text{loop gain}$$



$$s^2 + k_d s + k_p = \left(s^2 + k_d s\right)\left(1 + k_p\left(\frac{1}{s^2 + k_d s}\right)\right)$$

k_d = constant

k_p = variable gain

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k_d = variable gain

$$s^2 + k_d s + k_p = \left(s^2 + k_p\right)\left(1 + k_d\left(\frac{s}{s^2 + k_p}\right)\right)$$

$k_d = ck_p$

$k_p (\text{or } k_d)$ = variable gain

$$s^2 + k_d s + k_p = s^2\left(1 + k_d\left(\frac{s + k_p/k_d}{s^2}\right)\right) = s^2\left(1 + k_d\left(\frac{s + c^{-1}}{s^2}\right)\right)$$

$$s^2 + k_d s + k_p = s^2\left(1 + k_p\left(\frac{cs + 1}{s^2}\right)\right)$$

Multiple Gains

- **Choose one variable gain and fix relationship between all other gains and the variable gain.**
- **Write the characteristic equation (i.e., numerator of $1+L(s)$) as $M(s)(1+kF(s))$ where k =variable gain, $M(s)$, $F(s)$ are known transfer functions.**

From now on, we will only consider the standard form: $1+k\,F(s)$ where $F(s)$ is given.

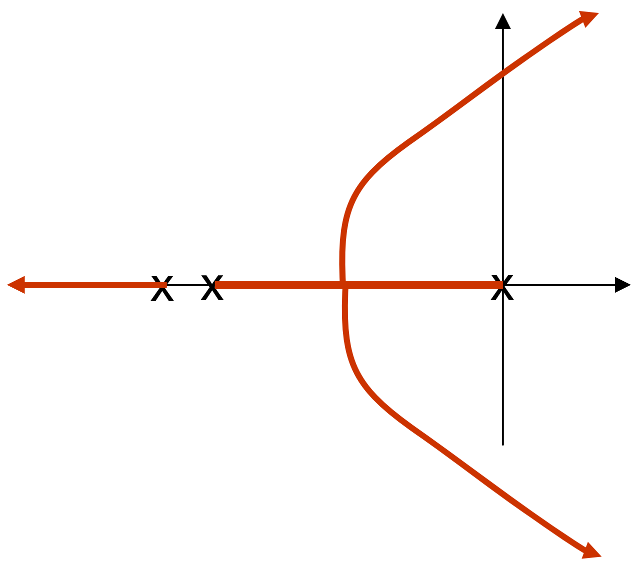
Example: PID with fixed PD

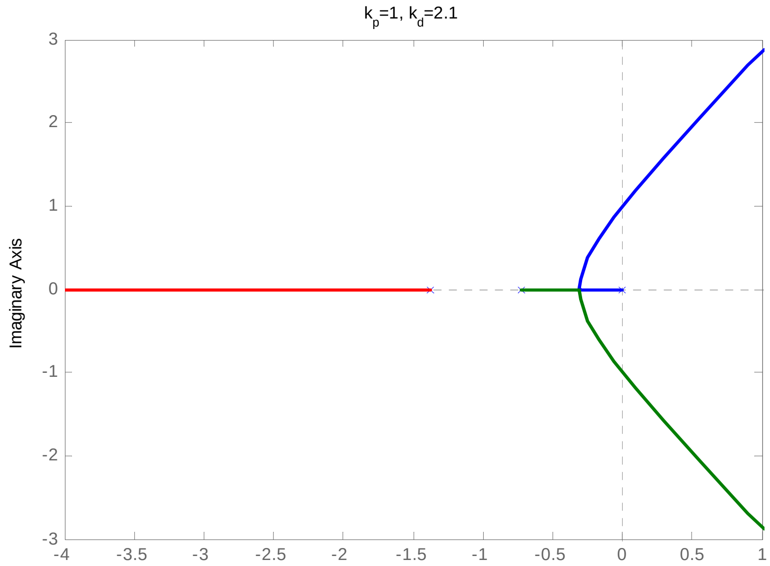
Return difference $$1+\frac{\left(k_p+k_d s+k_i/s\right)}{s^2}$$

Factor out the gain of interest

$$1+\frac{\left(k_p+k_d s+k_i/s\right)}{s^2}=1+\frac{\left(k_p+k_d s\right)}{s^2}+\frac{k_i}{s^3}$$

$$=\frac{\left(k_p+k_d s+s^2\right)}{s^2}+\frac{k_i}{s^3}=\frac{\left(k_p+k_d s+s^2\right)}{s^2}\left(1+\frac{k_i}{s\left(k_p+k_d s+s^2\right)}\right)$$





Rules for Root Locus

Write *F(s)=b(s)/a(s)* (convention: leading coefficient of *b(s)* is always positive).

$$1+k\frac{-s+1}{s^2+4} \rightarrow 1+\underbrace{(-k)}_{\text{new } k}\frac{s-1}{s^2+4}$$

$$1+kF(s)=1+k\frac{b(s)}{a(s)}=\frac{a(s)+kb(s)}{a(s)}$$

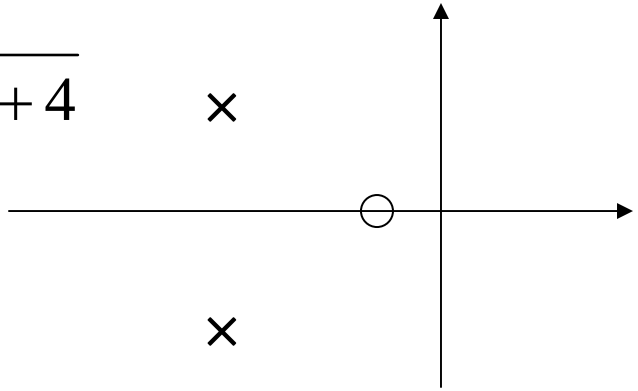
$$\text{characteristic polynomial}: a(s)+kb(s)$$

$$\text{when } k=0, \text{closed loop poles} = \text{open loop poles}$$

Steps for Drawing Root Locus

Step 1: Draw on the complex plane open loop poles and zeros (poles/zeros of *F(s)*, roots of *a(s)* and *b(s)*). use **x** for poles and **o** for zeros.

$$1+k\frac{s+1}{(s+2)^2+4}$$



Property of Root Locus

$$1+kF(s)=0 \Leftrightarrow \underbrace{F(s)}_{complex} = -\frac{1}{\underbrace{k}_{real}} \qquad \therefore \angle F(s) = \begin{cases} 180^o \pm l360^o & \text{if } k>0 \\ 0^o \pm l360^o & \text{if } k<0 \end{cases}$$

Terminology:

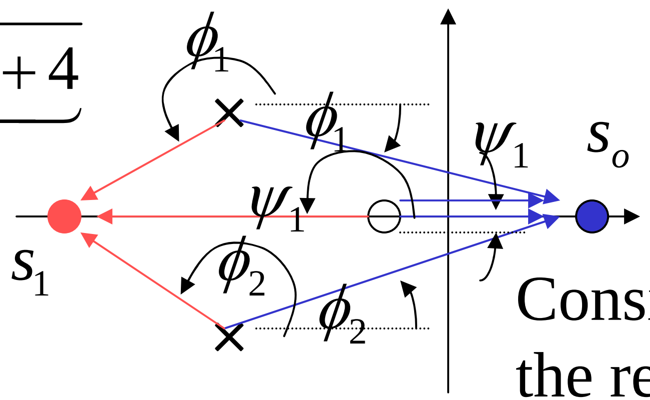
$k>0: 180^o$ locus

$k<0: 0^o$ locus

This means: In order for a complex number *s* to be on the 180^o locus, phase of *F(s)* = $180^o \pm l\, 360^o$; *s* is on the 0^o locus if phase of *F(s)*= $l\, 360^o$.

Property of Root Locus

$$1+k\underbrace{\frac{s+1}{(s+2)^2+4}}_{F(s)}$$



Consider any s_o on the real axis.

$$\angle F(s_o) = \psi_1 - \phi_1 - \phi_2 = 0^o$$

Therefore, s_o is on the 0^o locus.

When is s_o on the 180^o locus?

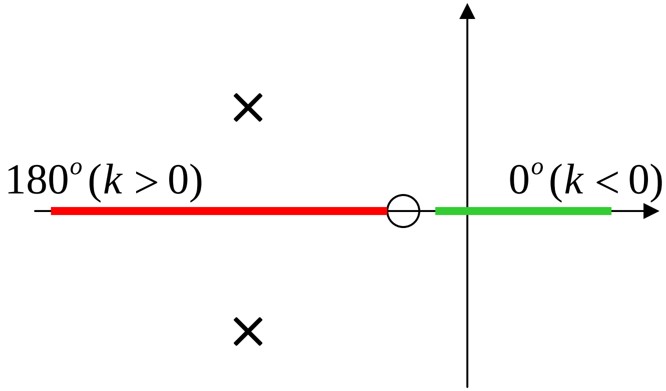
When is s_o on the 0^o locus?

$$\angle F(s_o) = \psi_1 - \phi_1 - \phi_2 = 180^o$$

Therefore, s_1 is on the 180^o locus.

Steps for Drawing Root Locus

Step 2: Consider any *s* on the real axis. If there is an ***odd*** number of ***real*** poles and zeros to the right of *s*, then *s* is on the 180° root locus. Otherwise, *s* is on the 0° root locus.



Steps for Drawing Root Locus

Step 3: Asymptotic root loci. Consider $k \to \infty$ and $k \to -\infty$.

$$1 + kF(s) = 1 + k\frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

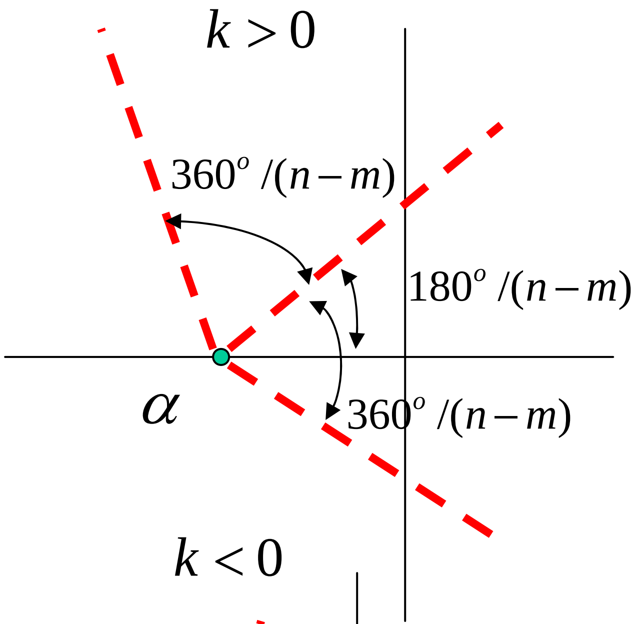
As $k \to \pm\infty, 1 + kF(s) = 0 \Rightarrow F(s) = -1/k \Rightarrow F(s) \to 0$

Therefore, m closed loop poles $\to$ m zeros of $F(s)$
n-m closed loop poles $\to \infty$. How do these poles approach ∞?

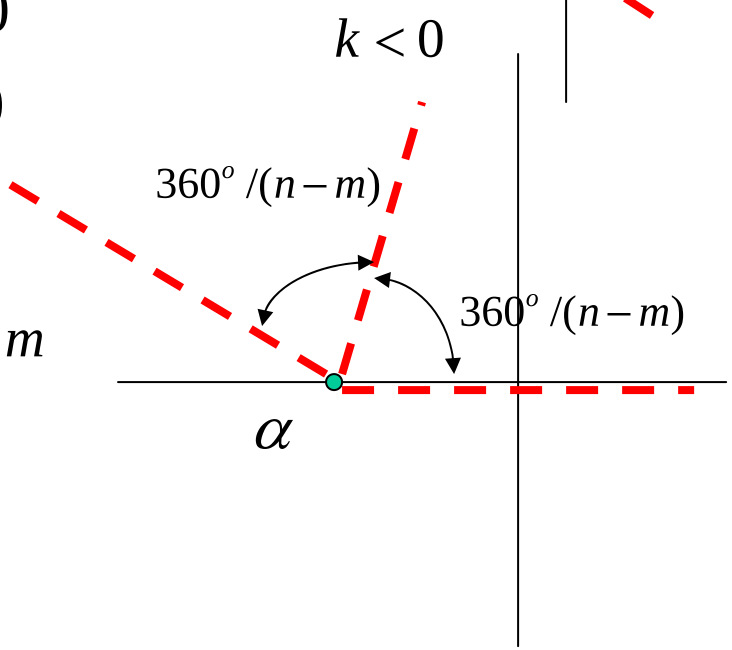
Steps for Drawing Root Locus

For large value of $s: F(s) \approx \dfrac{1}{(s-\alpha)^{n-m}}$

$$\alpha = \frac{\overbrace{\sum p_i}^{\text{sum of poles}} - \overbrace{\sum z_i}^{\text{sum of zeros}}}{n-m}$$



Recall: $\angle F(s) = \begin{cases} 180^o \pm l360^o & \text{if } k>0 \\ 0^o \pm l360^o & \text{if } k<0 \end{cases}$

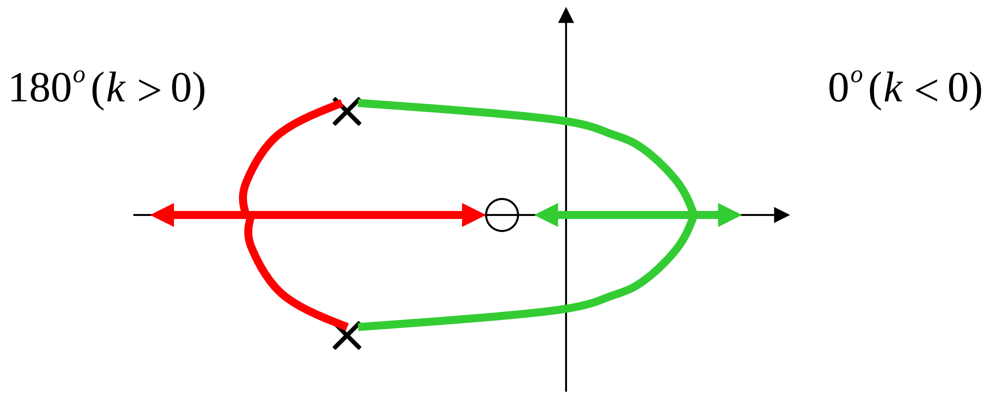


$$k>0: \angle(s-\alpha) = \frac{180^o + l360^o}{n-m}, l = 1, \cdots, n-m$$

$$k<0: \angle(s-\alpha) = \frac{-l360^o}{n-m}, l = 1, \cdots, n-m$$

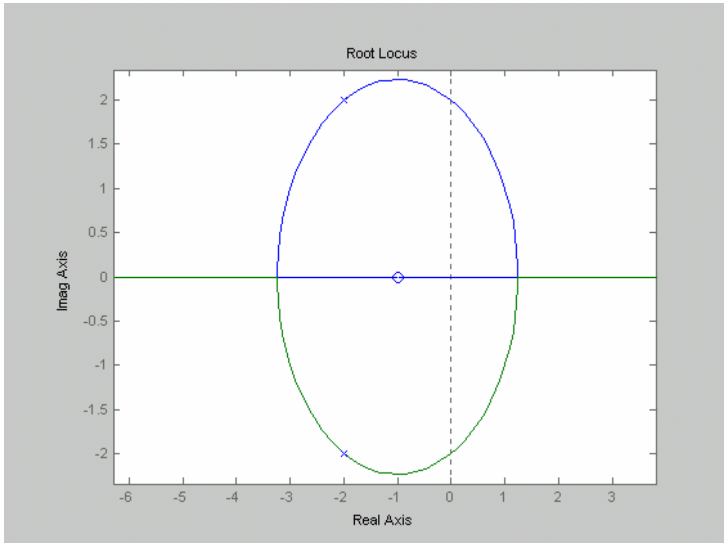
Example

$$1+k\frac{s+1}{(s+2)^2+4}$$



$$m=1, n=2, n-m=1$$

$$\alpha=-3$$

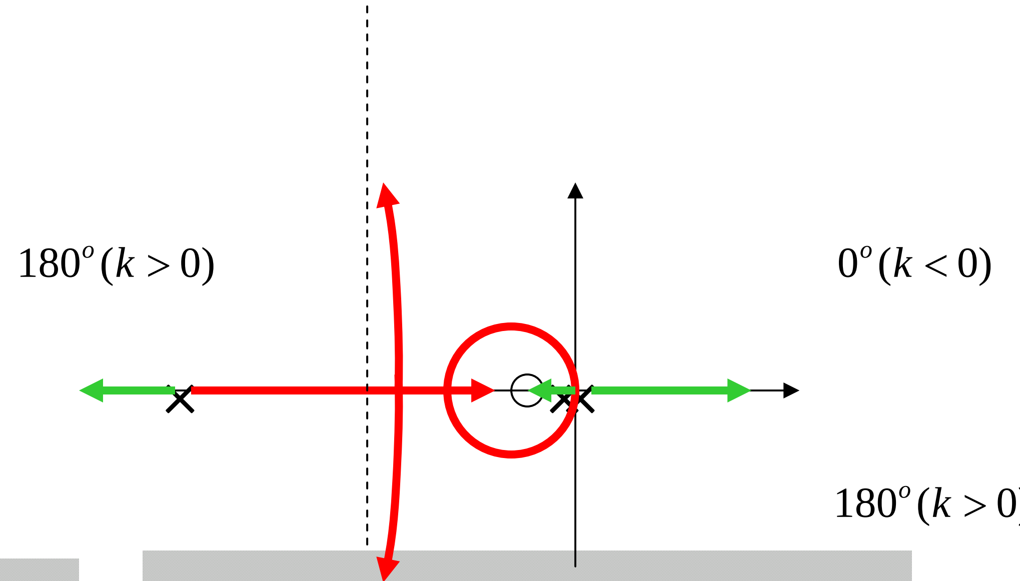


Example

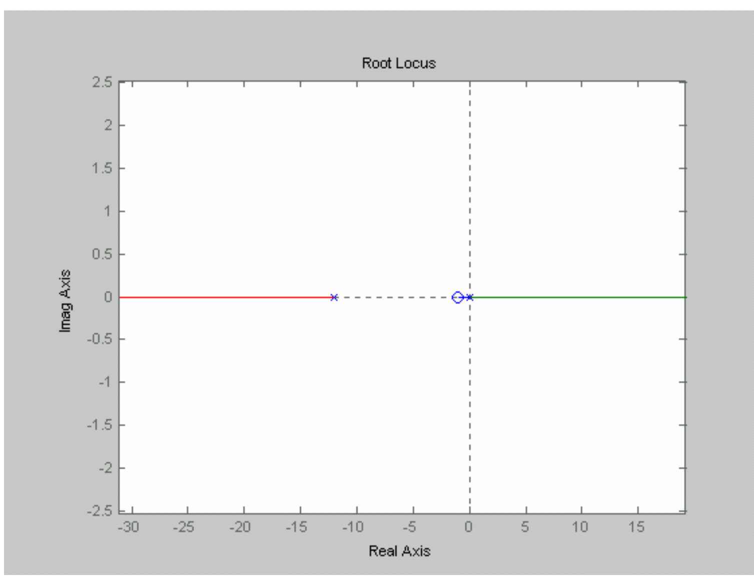
$$1 + k\frac{s+1}{s^2(s+12)}$$

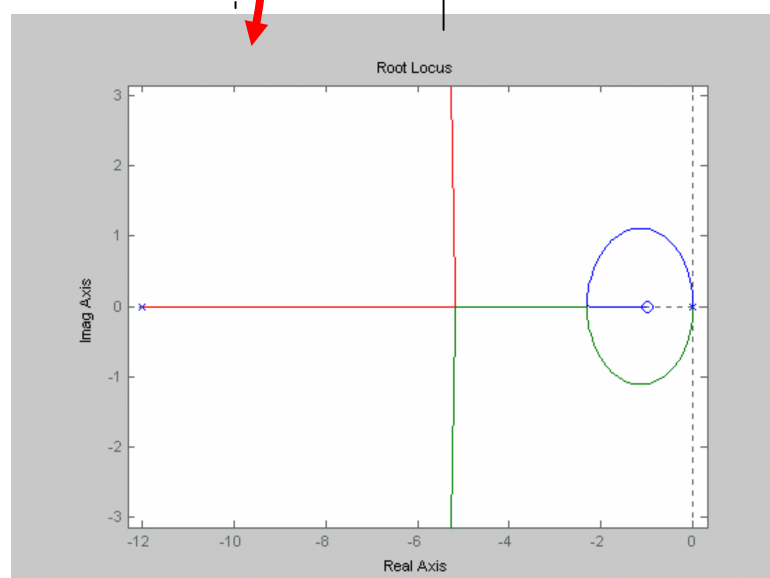
$$m = 1, n = 3, n - m = 2$$

$$\alpha = -11/2$$



$0^o(k<0)$

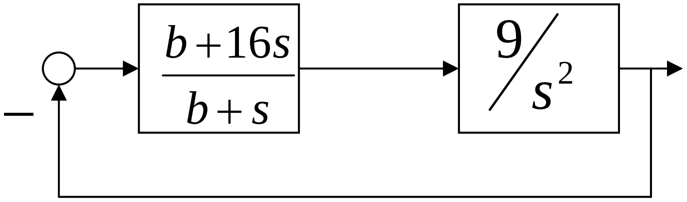




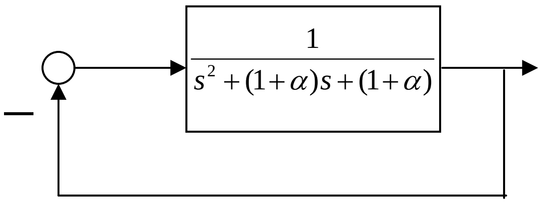
Today's Exercise

- sketch the complete root locus of the following systems first by hand and then compare with the MATLAB plot (use the rlocus command)

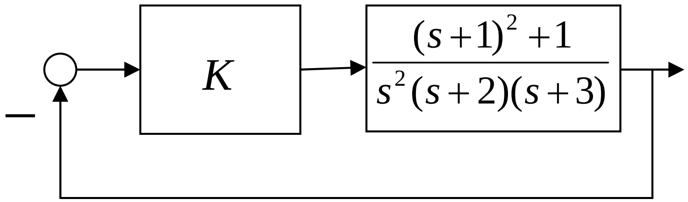
1.



3.



2.



Model for Two-link Arm

potential energy: $V = 0$ (gravity-free)

$$M(\theta_1,\theta_2) = \frac{1}{2}\begin{bmatrix} I_1 + I_{m_1}N_1^2 + m_1 l_{g_1}^2 + I_2 + m_2(l_1^2 + l_{g_2}^2 + 2l_1 l_{g_2}\mathrm{c}_2) & I_2 + m_2 l_{g_2}^2 + m_2 l_1 l_{g_2}\mathrm{c}_2 \\ I_2 + m_2 l_{g_2}^2 + m_2 l_1 l_{g_2}\mathrm{c}_2 & I_2 + I_{m_2}N_2^2 + m_2 l_{g_2}^2 \end{bmatrix}$$

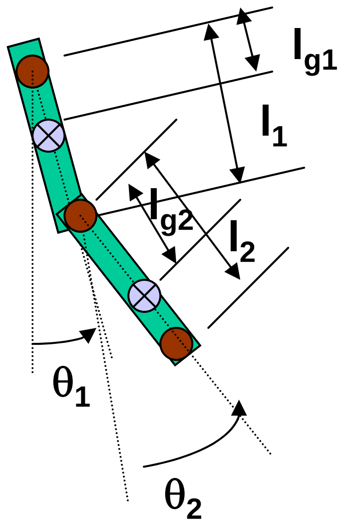
Lagrangian Method:

$$L = K - V$$

$$K = \frac{1}{2}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}^T M(\theta_1,\theta_2)\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = F$$

$$q = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$$



Equation of Motion:

$$M(q)\ddot q + C(q,\dot q)\dot q + N(\dot q) = u$$

Equilibrium and Linearization

Equilibrium (for zero input) : Any constant *q*.

Linearize about q_d : $$M(q_d)\ddot{q} + D\dot{q} = u$$

State Space Description

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$x_1 = x,\ x_2 = \dot{x}$$

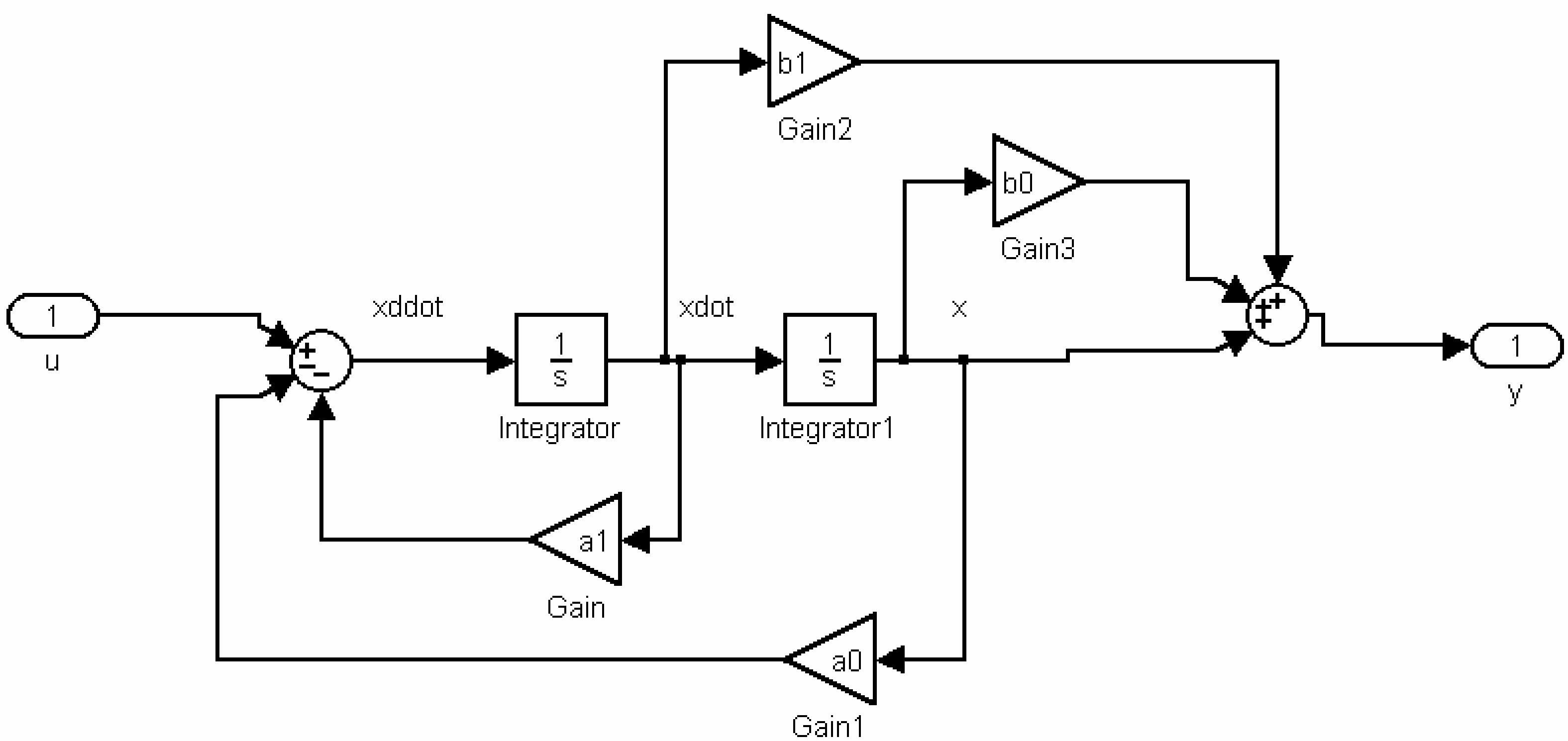
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a_0 x_1 - a_1 x_2 + u$$

$$y = b_0 x_1 + b_1 x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$



State Space Description

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} \cdot u$$

coefficients of denominator of *G(s)*

coefficients of numerator of *G(s)*

Controllable Canonical Form:

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{bmatrix}$$

Properties

- **poles of *G(s)* = eigenvalues of *A***
- **(A,B,C,D) is not unique for a given *G(s)***
- **MATLAB tools:**

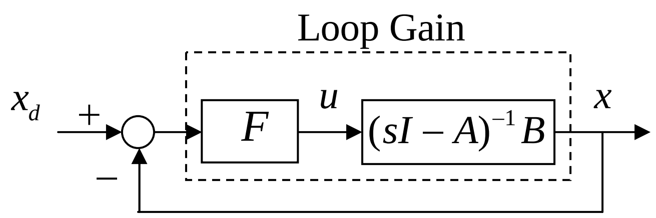
```
G=ss(A,B,C,D);[A,B,C,D]=ssdata(G);
```

Full State Feedback

- First assume that the full state vector *x* can be measured and be used for feedback control (*y*=*x*).
- Full state feedback control law: *u*=-*Fx*
- Closed loop system matrix: *A*-*BF*
- Closed loop poles: eig(*A*-*BF*)

$$\dot{x} = Ax + Bu$$
$$= (A - BF)x$$

Choice of Feedback Gain



- Choose *F* to "place" closed loop poles, eig(*A-BF*), and satisfy other design objectives (loop shape, robustness).
- Where to place the poles? Again use second order system as a starting point. Evaluate loop shape and robustness and iterate on the pole location.

Pole Placement

How do we choose *F* to place the closed loop poles?

- **Consider (*A*,*B*) in controllable canonical form:**

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- **Coefficients of det(sI-(*A*-*BF*)) (closed loop characteristic polynomial) are given by**

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

$$A - BF = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 - f_1 & -a_1 - f_2 & \cdots & -a_{n-1} - f_n \end{bmatrix}$$

Pole Placement

- **Choose *F* based on the desired closed loop characteristic polynomials.**

$$f_1 = a_{cl_0} - a_0, f_2 = a_{cl_1} - a_1, \cdots, f_n = a_{cl_{n-1}} - a_{n-1}$$

Example: $G(s) = 1/(s^2 + 2.427s)$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2.427 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Desired poles at (-10,-15)

Desired closed loop characteristic poly:

$$s^2 + 25s + 150$$

$$F = [150 \quad 25 - 2.427] = \begin{bmatrix} 150 & 22.573 \end{bmatrix}$$

- **MATLAB tools: `place`, `acker`**