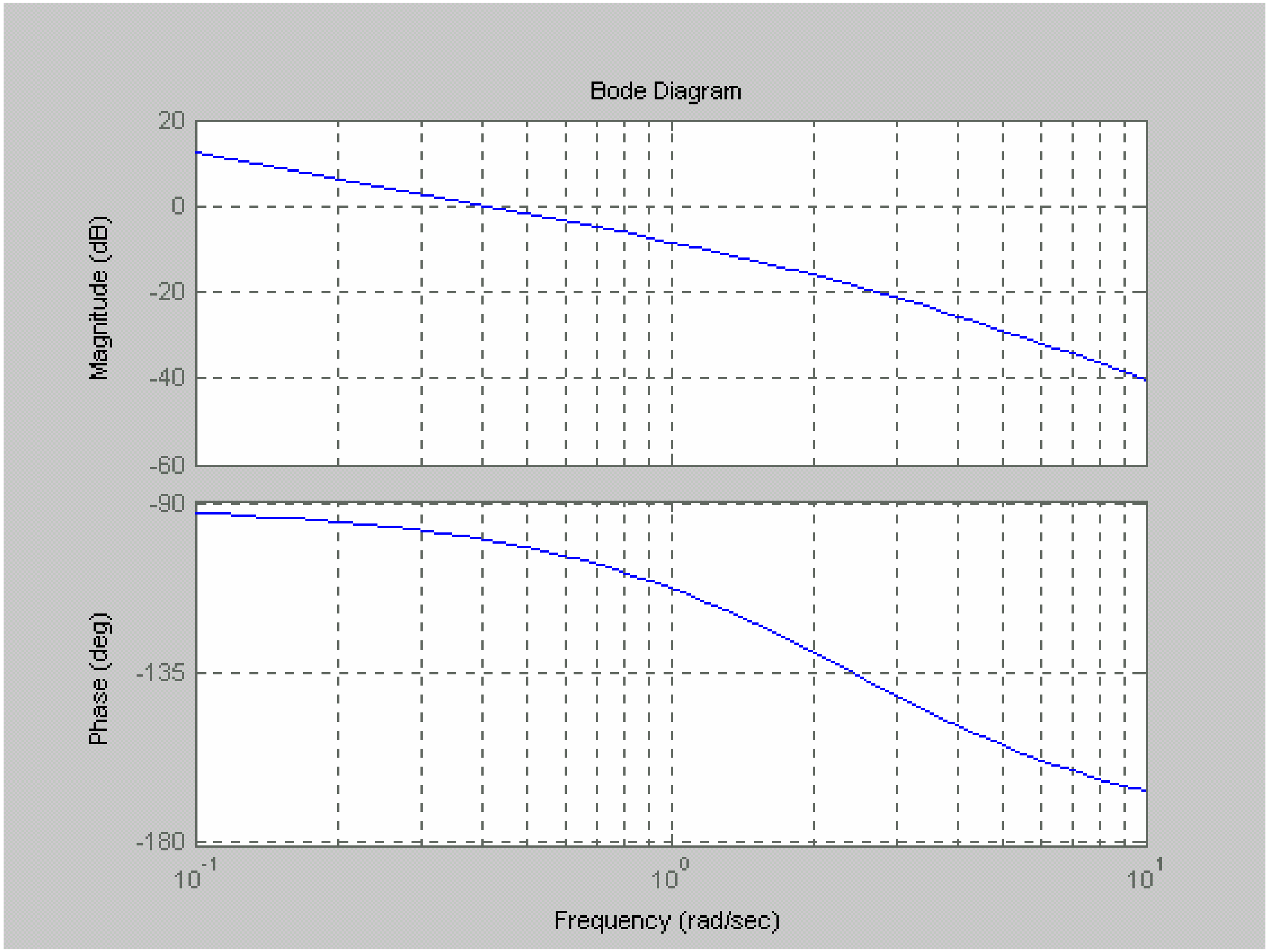
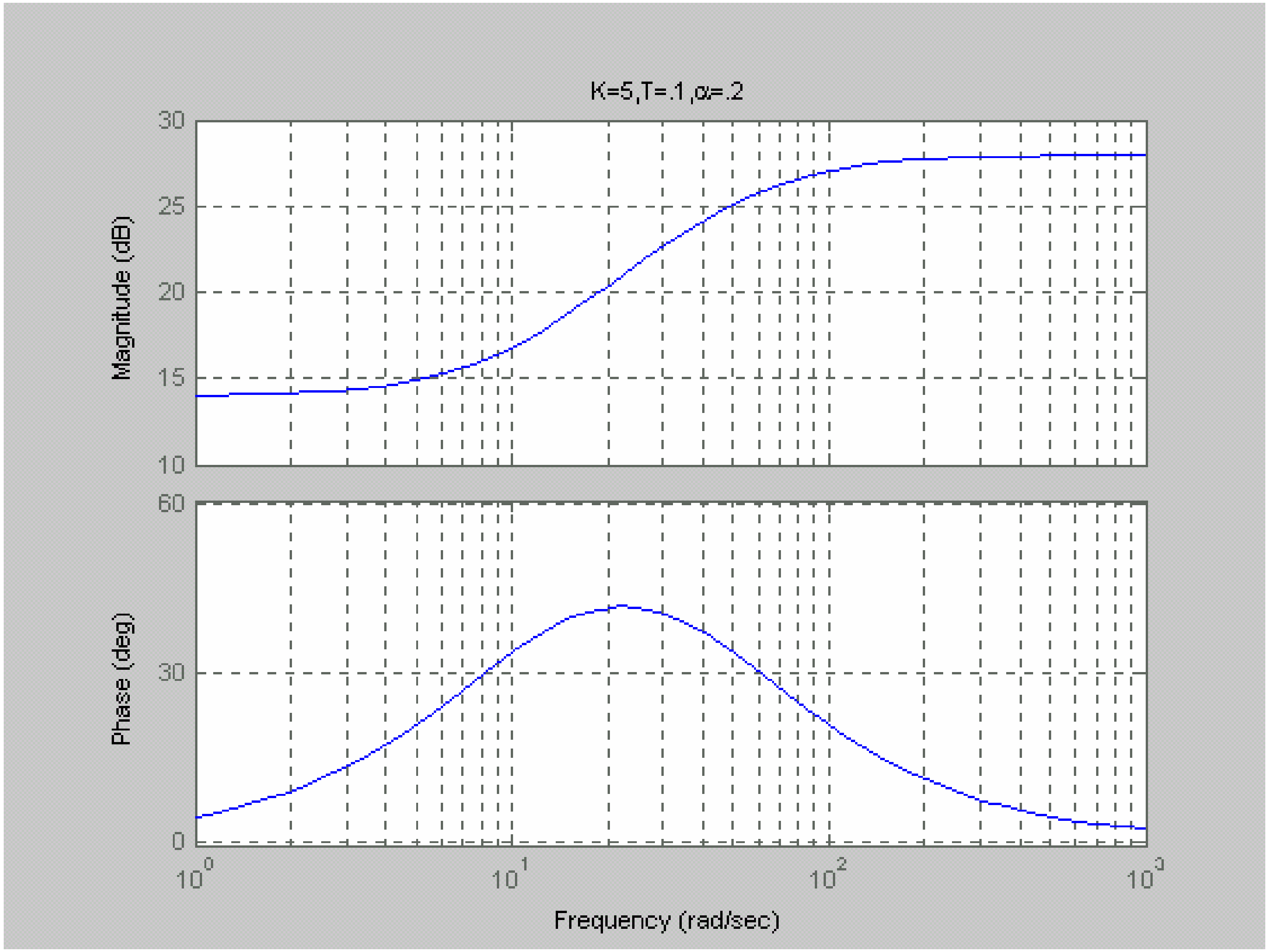
Bode Plot by Hand

$G(s)=1/(s^2+as)$



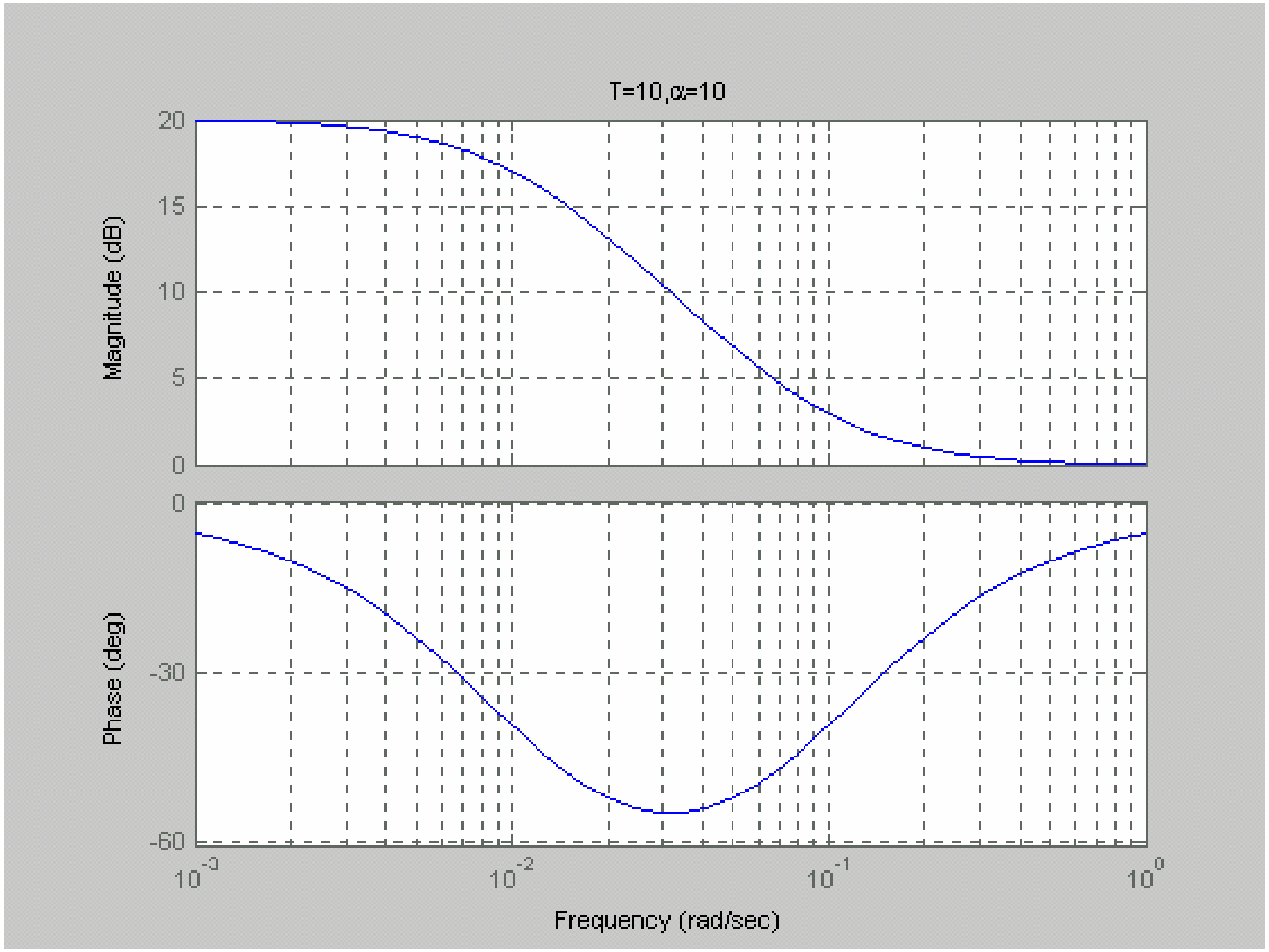
Bode Plot by Hand

lead filter: ***G(s)=K(Ts+1)/ (αTs+1)*** **α<1**

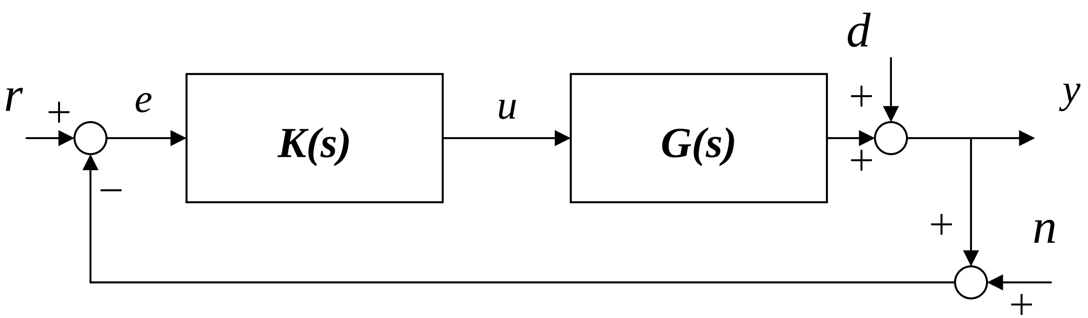


Bode Plot by Hand

lag filter: $G(s) = \alpha(Ts+1)/(\alpha Ts+1) \quad \alpha>1$



Sensitivity

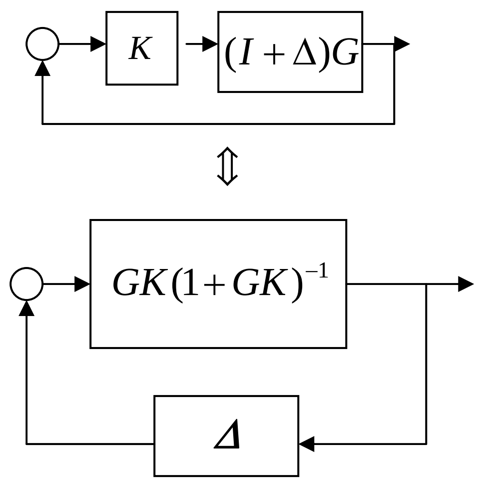


Reference to tracking error: $\hat{e} = \hat{r} - \hat{n} = \underbrace{(1+GK)^{-1}}_{\text{Sensitivity Function } S(s)} \hat{r}$

Sensor noise to output: $\hat{y} = \underbrace{GK(1+GK)^{-1}}_{\text{Complementary Sensitivity Function } T(s)} (-\hat{n})$

Reference to control effort: $\hat{u} = K(1+GK)^{-1}\hat{r}$

Actuator disturbance to output: $\hat{y} = (1+GK)^{-1}\hat{d}$

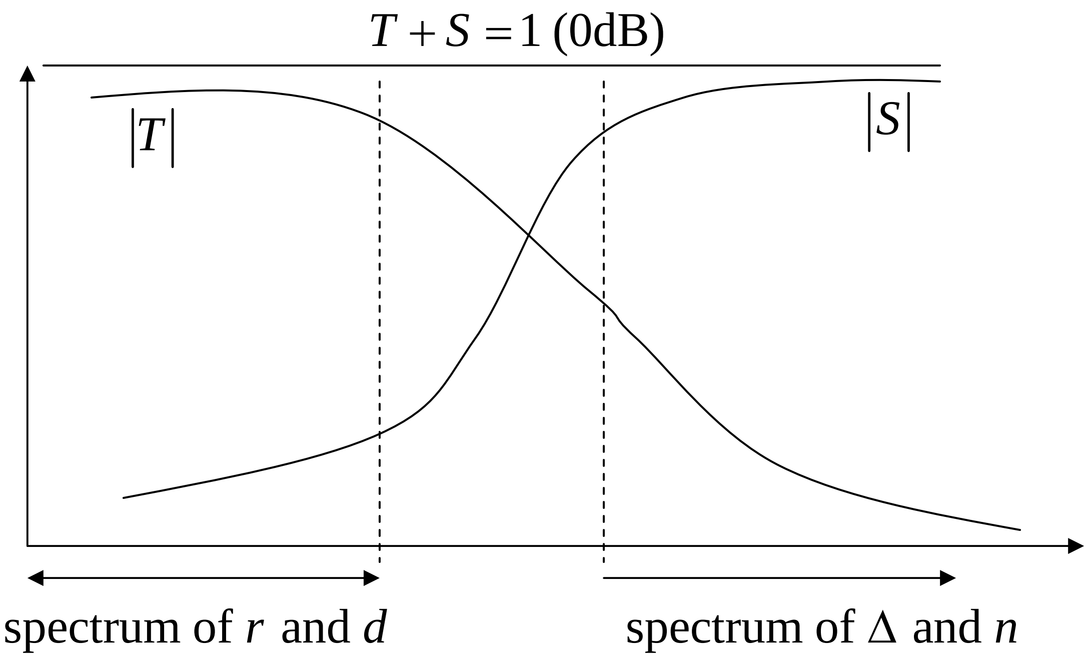


Control objective from loop shaping point of view:

Keep S small over bandwidth of r and d, keep T small over bandwidth of n and Δ, keep KS small over bandwidth of r (to avoid excessive control) **while maintaining closed loop stability.**

Sensitivity

However ... $S+T=1$, so we cannot make both of them small.

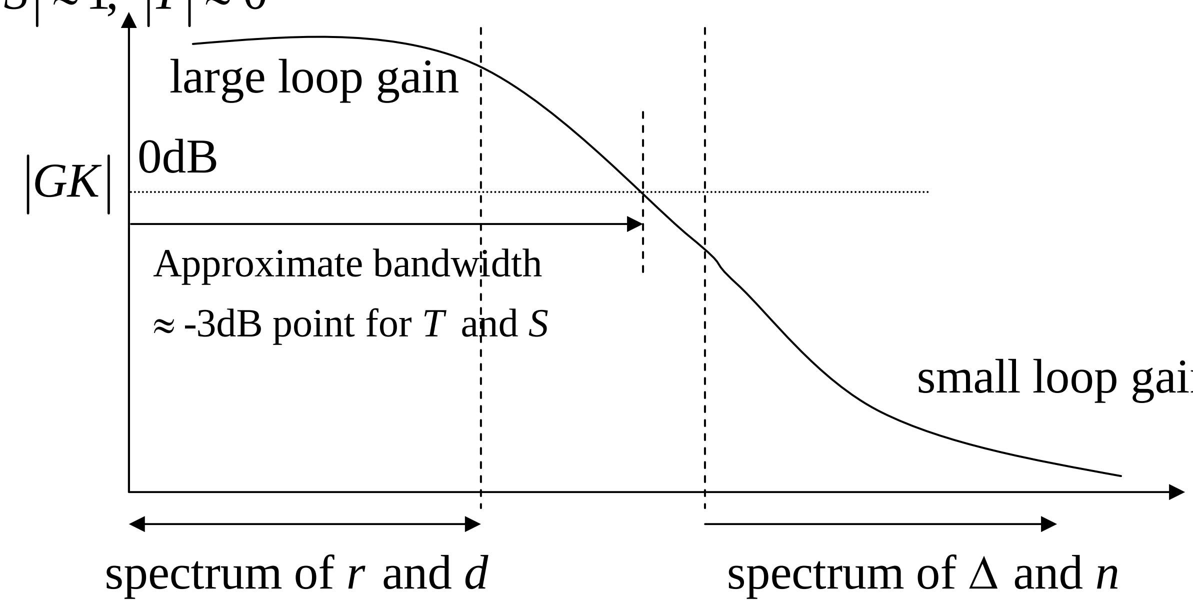


Design Through Loop Gain

Design K to shape S and T is difficult since the relationship is nonlinear. It is much easier to design based on loop gain GK.

Note that $|GK| \gg 1 \Rightarrow |T| \approx 1,\ |S| \approx 0$

$|GK| \ll 1 \Rightarrow |S| \approx 1,\ |T| \approx 0$



We will show next time how to incorporate stability and robustness directly into loop gain. This will allow us to perform complete control design through loop shaping.

Steady State Error

If input r is a step, steady state error is $S(0)=1/(1+G(0)K(0))$. Therefore, large DC (zero frequency) gain ➔ small steady state error.

Similarly, if we would like to track a sinusoid (as in Project 1) with frequency ω_o, we should choose $K(s)$ so that $S(j\omega_o)=1/(1+G(j\omega_o)K(j\omega_o))$ is small ($G(j\omega_o)K(j\omega_o)$) is large.

We also want to avoid large controller gain, since it would lead to large bandwidth and make the closed loop system less robust and more susceptible to sensor noise.

Large gain also leads to large control effort since $K(s)S(s)$ would be large.

MATLAB Commands

```
%sensitivity and complementary sensitivity functions
loop_gain=G*K;
T=feedback(G*K,1);
S=feedback(1,G*K);
KS=feedback(K,G);
```

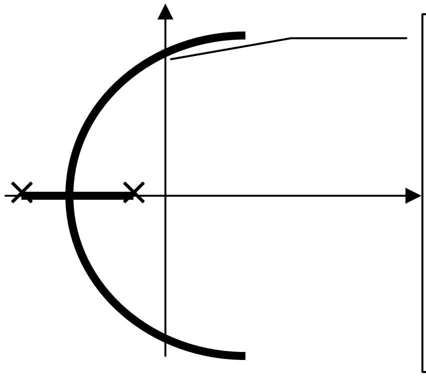
Summary

Control design through shaping of the loop gain *GK*: keep *GK* large in the spectra of ref input & disturbance, keep *GK* small in the spectra of model uncertainty & sensor noise, keep *K* small for small control effort).

We will next derive stability and robustness conditions in the frequency domain.

Stability of CL System

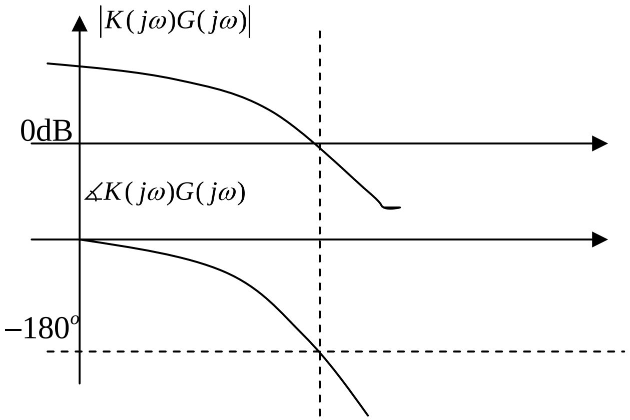
Consider an open loop stable system that becomes unstable with large gain:



At the point of instability,

$$|K(j\omega)G(j\omega)| = 1$$

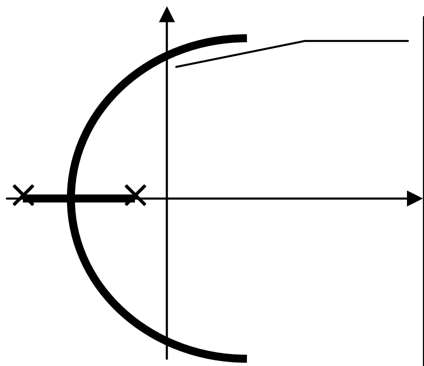
$$\measuredangle K(j\omega)G(j\omega) = \pm 180^o$$



Closed loop poles must satisfy:

$$(1 + K(s)G(s)) = 0 \Rightarrow |K(s)G(s)| = 1 \text{ and } \measuredangle K(s)G(s) = \pm 180^o$$

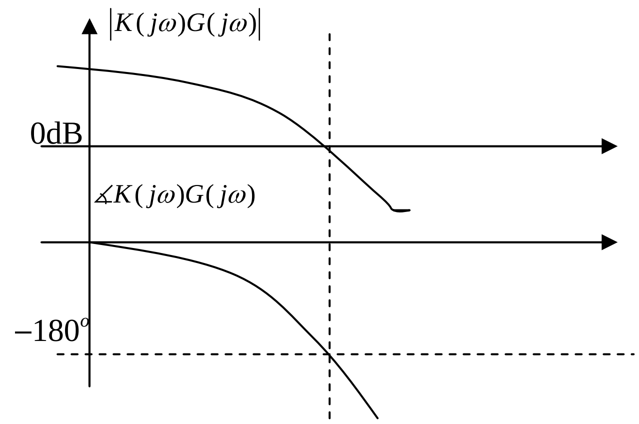
Stability Margins



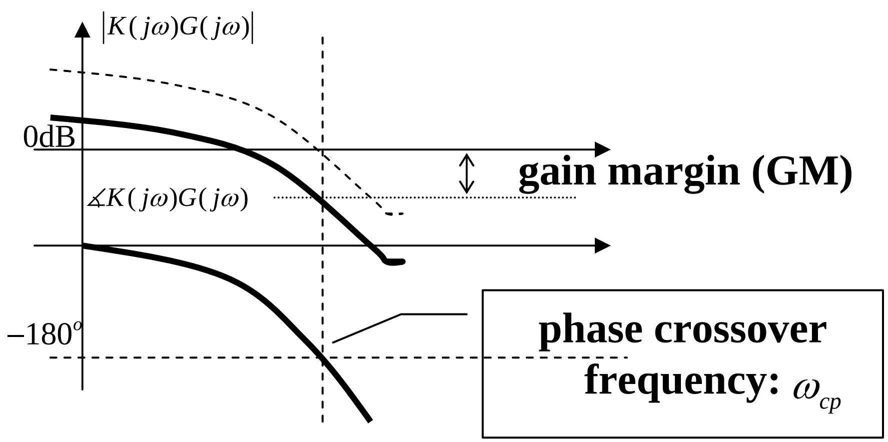
At the point of instability,

$$|K(j\omega)G(j\omega)| = 1$$

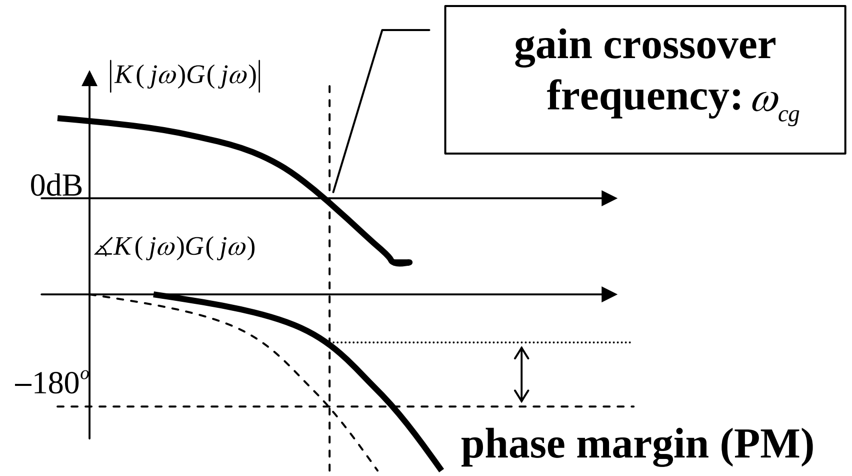
$$\measuredangle K(j\omega)G(j\omega) = \pm 180^o$$



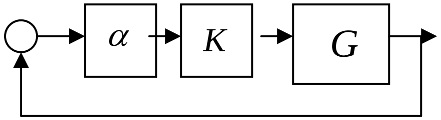
If actual gain is smaller:



If actual phase is closer to 0º:



Stability Margins



Gain margin means the amount of gain variation (α is a positive real number) that can be tolerated. If $\alpha>1$, $(\alpha)_{dB}>0$; if $\alpha<1$, $(\alpha)_{dB}<0$.

Phase margin means the amount of phase variation (α is a phase shift: $\alpha=e^{j\phi}$) that can be tolerated.

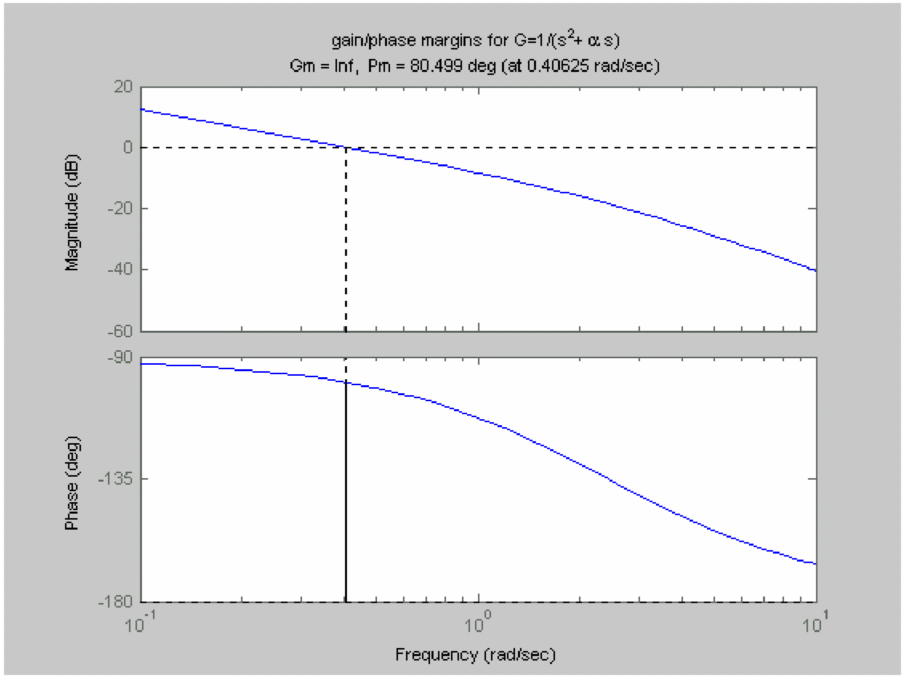
Stability Margins

What if there are multiple gain crossover and phase crossover points?

We will discuss Nyquist plot in the next lecture which will give us a definitive answer.

For now, use MATLAB `margin` command:

```
margin(sys);
```



Loop Shaping Perspective

If we can reduce the gain near the phase crossover, we can improve the gain margin (gain stabilization through gain roll-off).

If we can increase the phase (add phase lead) near the gain cross over, we can improve the phase margin (phase stabilization through lead compensation).

Unfortunately, gain and phase are not independent. Hence, changing gain ➔ changing phase. We'll see this later in Bode gain/phase formula.

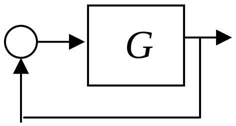
Relationship to Performance

How do we infer performance directly from the loop gain?

Based on the standard second order systems (with no zero), we have the following rules of thumb:

$$\zeta \approx \frac{\text{PM}}{100} \quad \text{(e.g., } 30^{\text{o}} \text{ PM} \approx 30\% \text{ damping)}$$

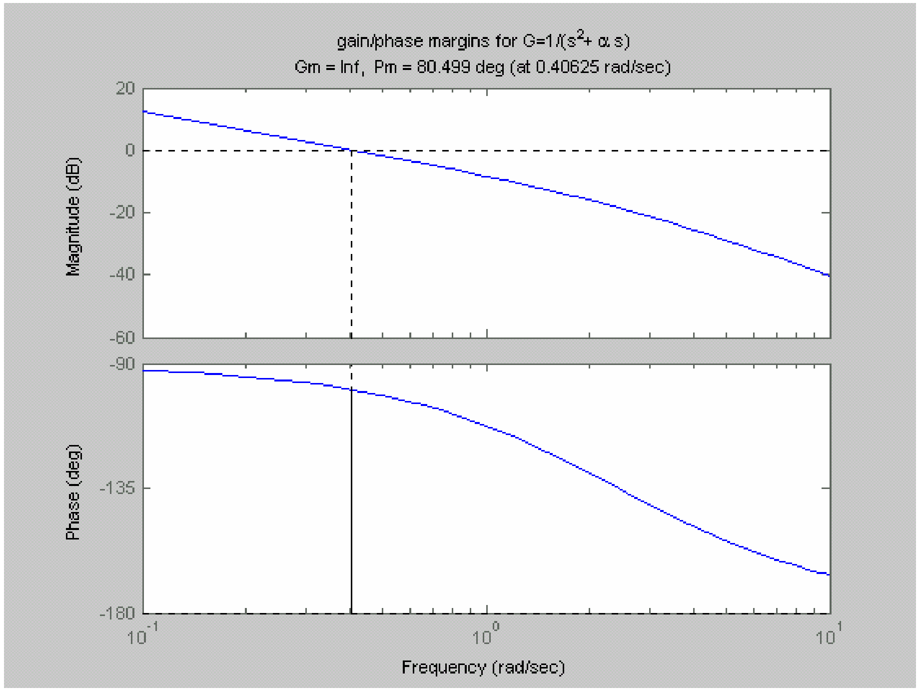
$$\omega_{\text{n}} \approx \omega_{cg} \quad \text{(gain cross over frequency, i.e., BW)}$$

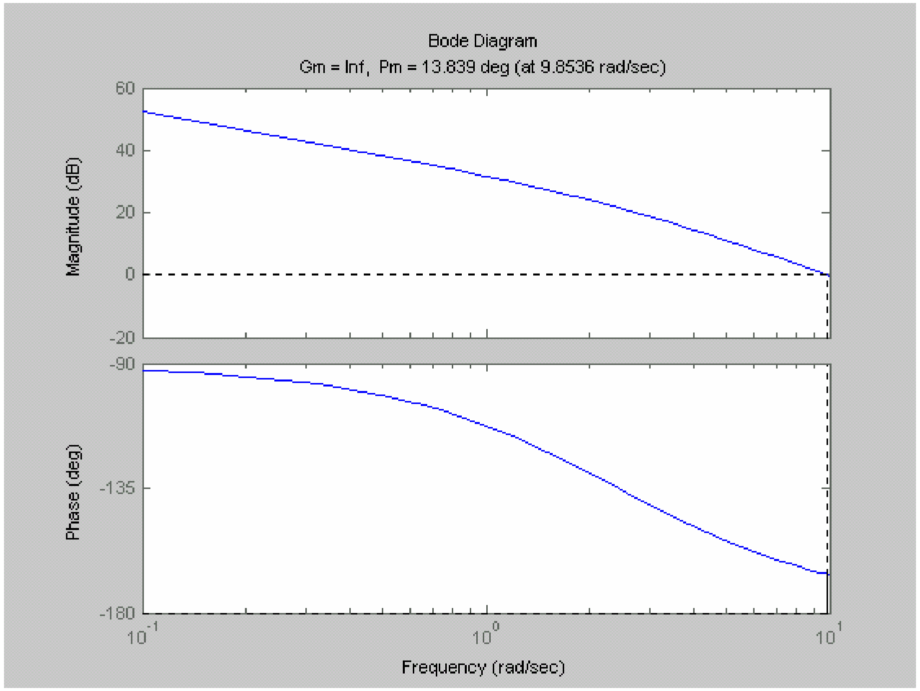


$$G(s) = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s}$$

$$G_{cl}(s) = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

Example



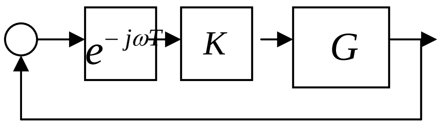


k_p=*1*, PM=80.5°, ω_{cg} = .41rad/sec

k_p=*100*, PM=13.8°, ω_{cg} = 9.85rad/sec

$$K(s)G(s) = \frac{k_p}{s^2 + as}$$

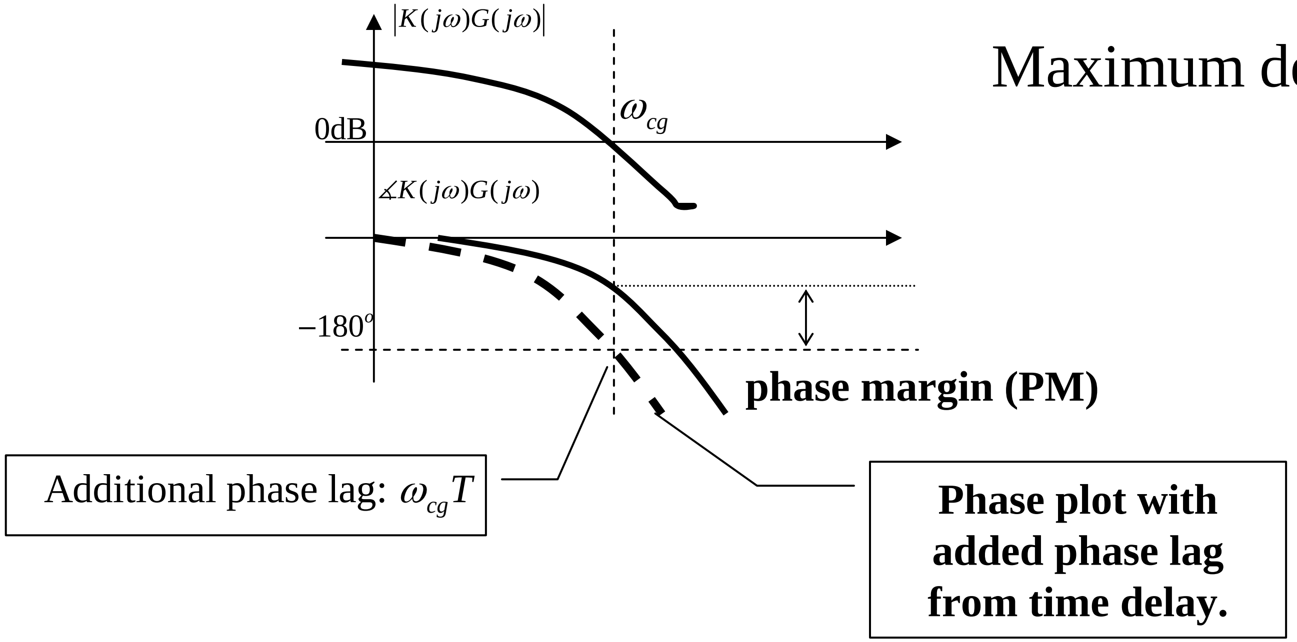
Time Delay



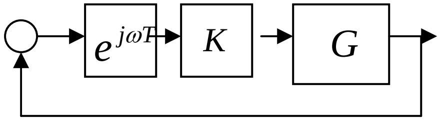
Time delay adds a phase shift of -ωT.

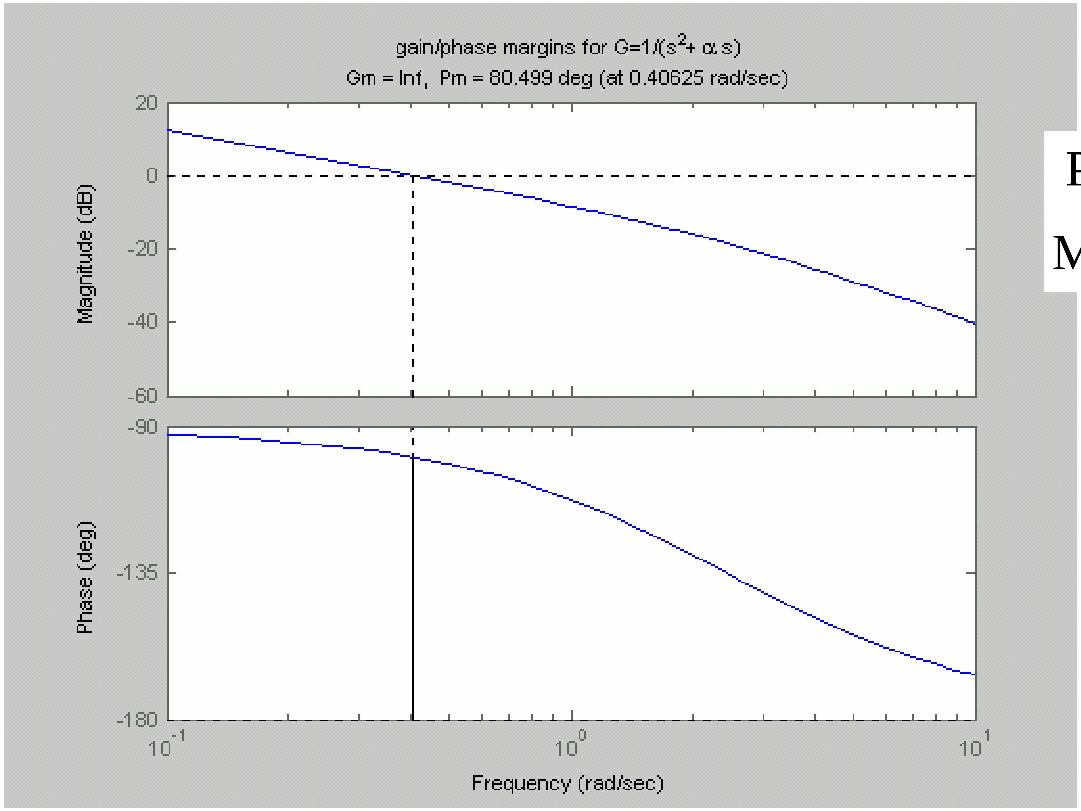
Boundary of stability: PM=$\omega_{cg}T$

Maximum delay: $T_{\max} = \dfrac{\text{PM}}{\omega_{cg}}$



Time Delay





$$\text{PM}=80.5^{\circ}=1.405\text{rad}, \omega_{cg}=.41\text{rad/sec}$$

$$\text{Maximum delay: } T_{\max}=3.45\text{sec}$$

```
G=tf(1,[1 Fv/Ic 0]);
T=3.46; % T=3.45
[n,d]=pade(3,T);Gd=tf(n,d);
max(real(pole(feedack(G*Gd,
1))))
```

```
T=3.46: 1.52e-5
T=3.45: -3.93e-4
```

Summary

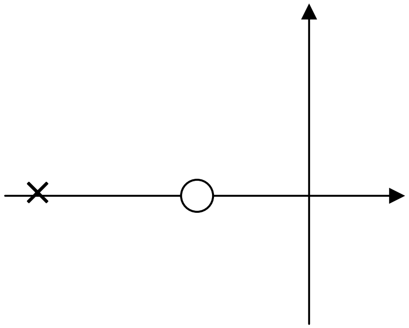
Frequency domain control design involves choosing $K(s)$ to achieve a loop gain $K(s)G(s)$ with the following attributes:

- large loop gain in spectra of r and d (e.g., for trajectory tracking and input disturbance rejection).
- small loop gain in spectra of n and Δ (e.g., for sensor noise rejection and unmodeled dynamics).
- small enough K to avoid actuator saturation
- adequate gain margin for gain robustness
- adequate phase margin for damping and time delay
- adequate bandwidth for speed of response

Lead Compensation

Lead filter can be used to add phase lead (improve phase margin) and increase bandwidth.

$$K_{lead}(s) = K\frac{Ts+1}{\alpha Ts+1}, \alpha < 1$$



Three design parameters: K, α, T

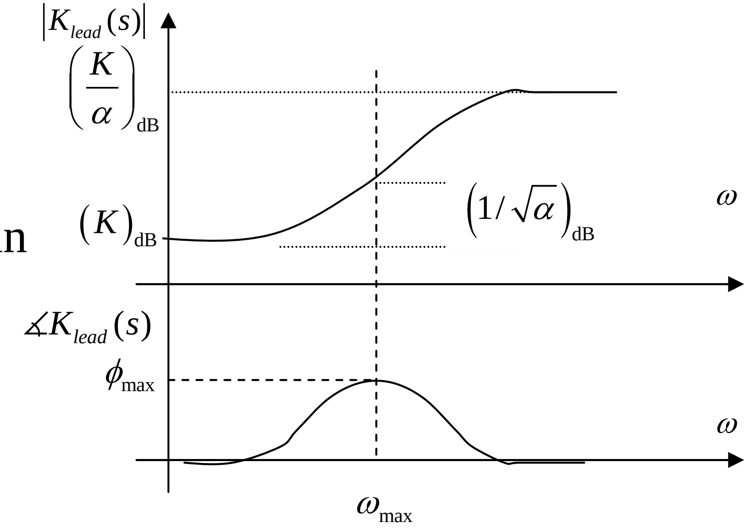
(i.e., overall gain, pole/zero locations)

key attributes:

max phase, location of max phase, high freq gain

$$\alpha = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$

$$\omega_{\max} = 1/T\sqrt{\alpha}$$



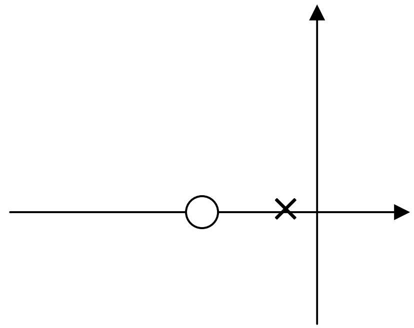
Lead Design Procedure

Given *G(s)*:

- **Determine open loop gain *K* to meet low freq gain requirement (*KG(0)*) and/or bandwidth requirement (BW *KG(s)* about ½ of desired closed loop BW). Gain crossover freq=ω_{cg}.**
- **Evaluate PM of *KG(s)*. Determine extra phase lead needed, set it to ϕ_{max} .**
- **Determine α. Find the new gain crossover freq ω_{cg1}** $KG(j\omega_{cg1}) = \left(\sqrt{\alpha}\right)_{\text{dB}}$
- **Let $\omega_{max} = \omega_{cg1}$ and solve for *T.***
- **Check PM, BW of $G(s)K_{lead}(s)$ and iterate if necessary.**
- **Check all other specifications, and iterate design; add more lead compensators if necessary.**

Lag Compensation

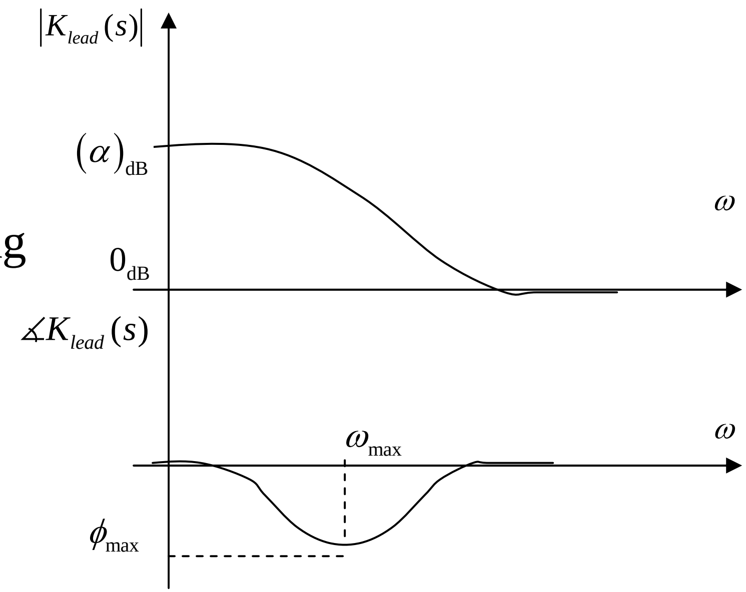
Lag filter can be used to boost DC gain (to reduce steady state error; but can reduce phase margin.



$$K_{lag}(s) = \alpha \frac{Ts+1}{\alpha Ts+1}, \alpha > 1$$

Two design parameters: α, T

(i.e., pole/zero locations)



key attributes:

low freq gain, max phase lag, location of max lag

$$\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}, \phi_{max} < 0$$

$$\omega_{max} = 1 / T\sqrt{\alpha}$$

Lag Design Procedure

Given *G(s)*:

- **Determine overall open loop gain *K* to meet PM requirement (without lag compensation).**
- **Determine α to achieve desired low frequency gain.**
- **Choose 1/*T* (zero location) to be 1 decade below gain crossover frequency of *KG(s)*.**
- **Check all other specifications, and iterate if necessary.**

Example

Given *G(s)=1/(2s+1)(s+1)(.5*s+1)*, design a lead compensator so that DC gain = 9 and PM > 25°

Lead Design Procedure

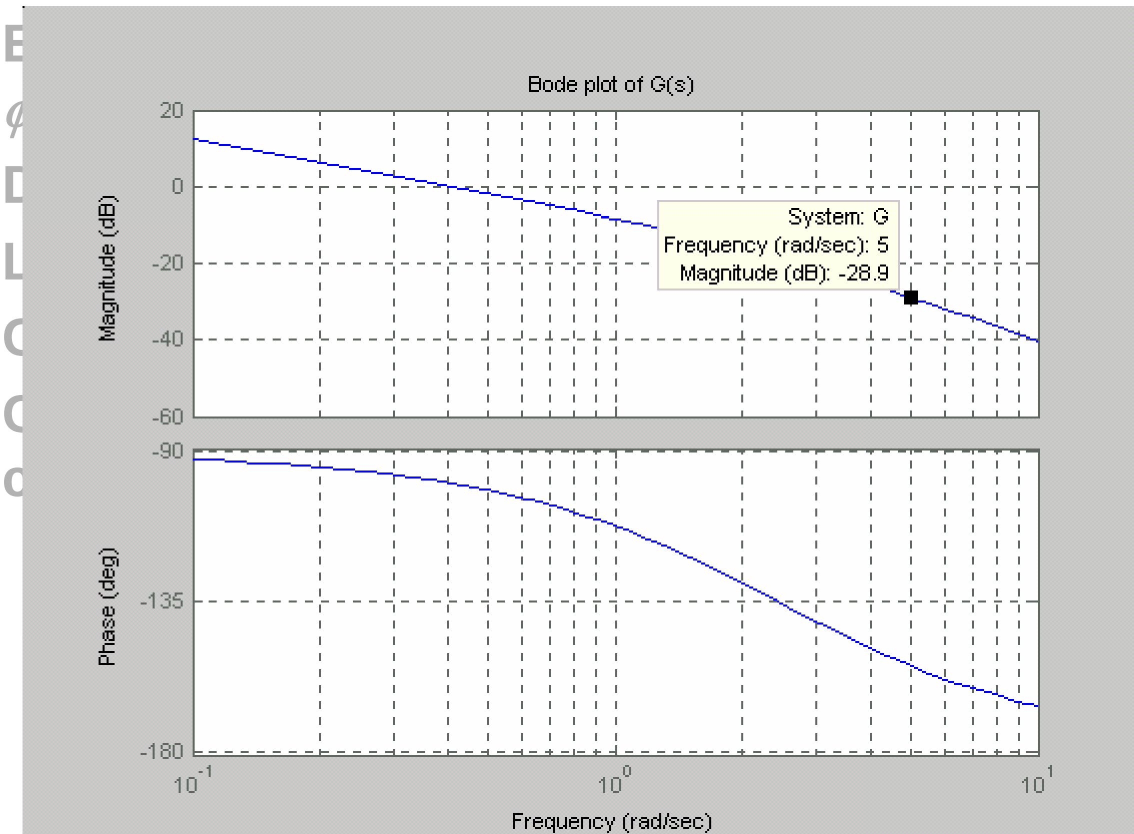
Given *G(s)*:

- **Determine open loop gain *K* to meet low freq gain requirement (*KG(0)*) and/or bandwidth requirement (BW *KG(s)* about ½ of desired closed loop BW). Gain crossover freq=ω_{cg}.**
- **Evaluate PM of *KG(s)*. Determine extra phase lead needed, set it to ϕ_{max} .**
- **Determine α. Find the new gain crossover freq ω_{cg1}** $KG(j\omega_{cg1}) = \left(\sqrt{\alpha}\right)_{\text{dB}}$
- **Let $\omega_{max} = \omega_{cg1}$ and solve for *T*.**
- **Check PM, BW of $G(s)K_{lead}(s)$ and iterate if necessary.**
- **Check all other specifications, and iterate design; add more lead compensators if necessary.**

Lead Design Procedure

Given *G(s)=1/(s²+as)*:

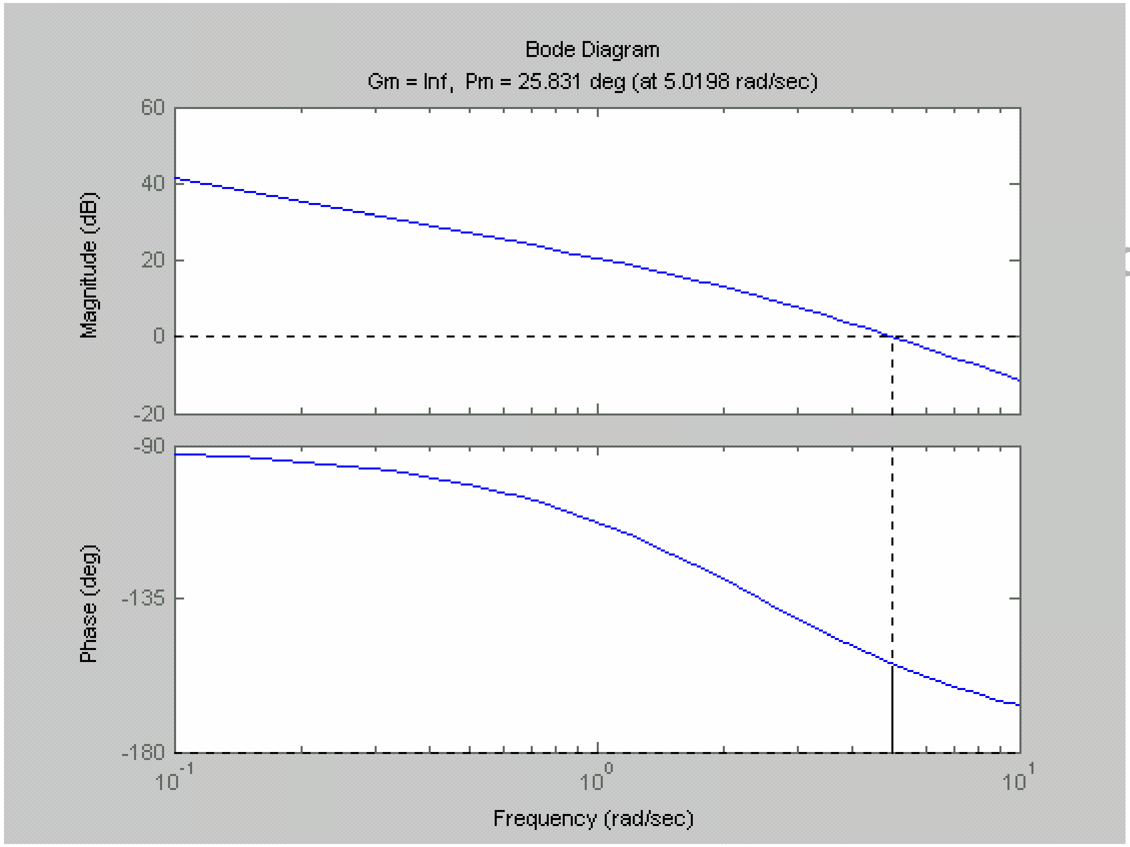
- **Determine open loop gain *K* to meet** low freq gain requirement (*KG(0)*) and/or **bandwidth requirement (BW *KG(s)* about ½ of desired closed loop BW). Gain crossover freq=ω_{cg}.**
- E… e lead needed, set it to φ…
- D… eq o…
- L…
- C… cessary.
- C… sign; add more lead c…



Choose K=28.9dB=28

Lead Design Procedure

- **Evaluate PM of *KG(s)*. Determine extra phase lead needed, set it to ϕ_{max}.**



fre

ne

des

Target PM=60deg, so we need 35deg from lead filter. Add 5 deg extra pad.

$$\phi_{max} = 39.2\deg = .684 \text{rad}$$

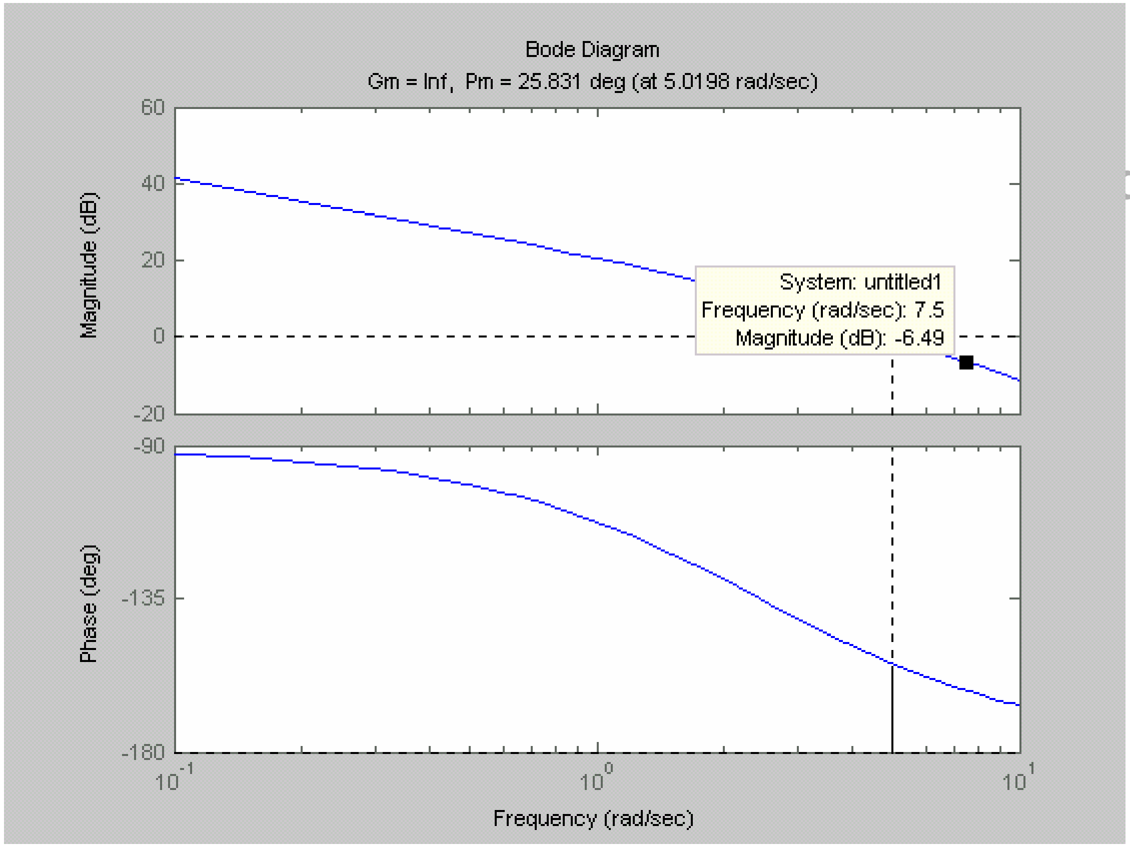
$$\alpha = .225$$

extra gain from lead filter at ω_{max}

$$= \left(\sqrt{\alpha}\right)_{dB} = 6.47 \text{dB}$$

Lead Design Procedure

- **Determine α. Find the new gain crossover freq ω_{cg1}**



New crossover freq @ 7.5rad/sec.

Substitute $\omega_{\max} = 7.5\text{rad/sec}$ into

$$T = \frac{1}{\omega_{\max}\sqrt{\alpha}}$$

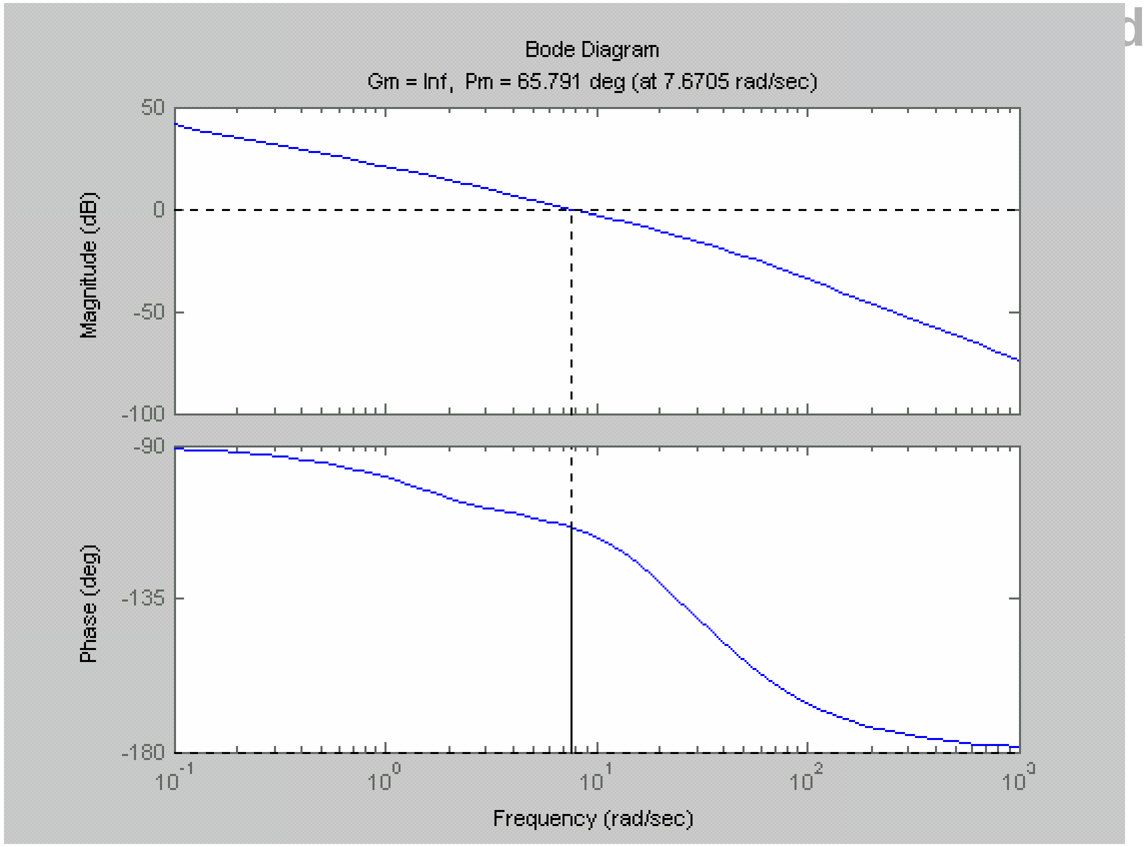
we get $T = .28$

Overall lead filter:

$$K_{lead}(s) = 124.2\frac{s+3.56}{s+15.79}$$

Lead Design Procedure

- **Check PM, BW of $G(s)K_{lead}(s)$ and iterate if necessary.**



Not quite meeting the spec, so iterate!

Change $\phi_{\max} = 50^o$ and

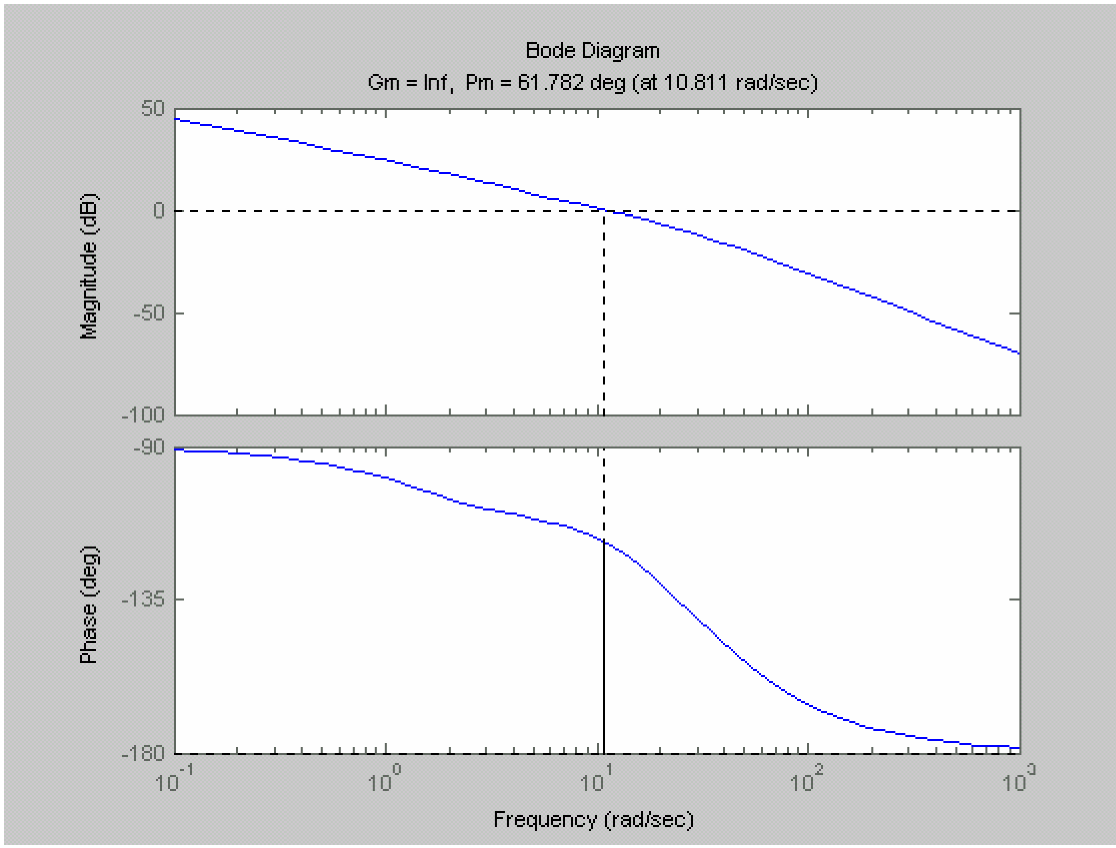
$\omega_{\max} = 10 \text{rad/sec}$

Overall lead filter:

$$K_{lead}(s) = 202.4 \frac{s + 3.72}{s + 26.89}$$

Lead Design Procedure

- **Check PM, BW of *G(s)K***

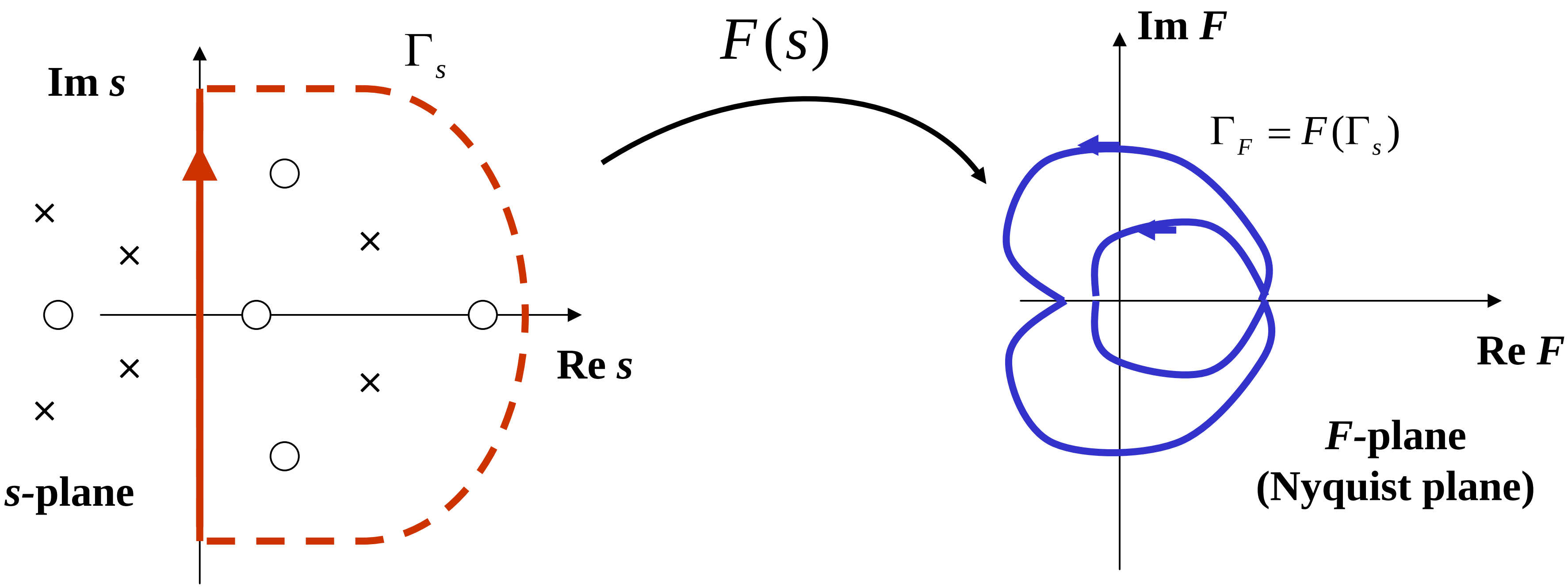


More PM than we need, so increase the gain a bit.

Overall lead filter:

$$K_{lead}(s) = 303.7\frac{s+3.72}{s+26.89}$$

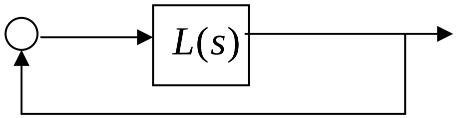
Principle of Argument



Let *Z*=# of zeros of *F(s)* enclosed by Γ, *P*=# of poles of *F(s)* enclosed by Γ.

Then # of counterclockwise (CCW) encirclement of origin of *F*-plane by *F*(Γ) = *P*-*Z*

Nyquist Stability Criterion



Choose:

- Γ **encloses the entire right half complex plane**
- ***F(s)=1+L(s)***

Then

- ***Z*** **= # of unstable zeros of *1+L(s)***

  = 0 if the closed loop system is stable
- ***P*** **= # of unstable poles of *1+L(s)***

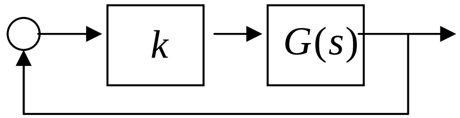
  = # of open loop unstable poles (# of unstable poles in *L(s)*)

From the Principle of Argument,

closed loop system is stable if and only if

# of CCW encirclement of the origin in the Nyquist plane by Γ_{1+L} **= # of open loop unstable poles**

Nyquist Stability Criterion



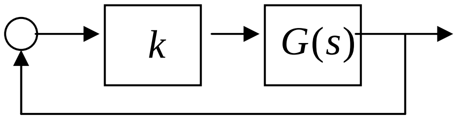
If $L(s) = k\,G(s)$, then

closed loop system is stable if and only if

\# of CCW encirclement of $(-1/k, 0)$ in the Nyquist plane by Γ_G = # of open loop unstable poles

Example

$$G(s) = \frac{s+2}{s^2 - 1}$$



$$G(j\omega) = \frac{j\omega + 2}{-\omega^2 - 1}$$

$$\operatorname{Re} G(j\omega) = -\frac{2}{1+\omega^2}$$

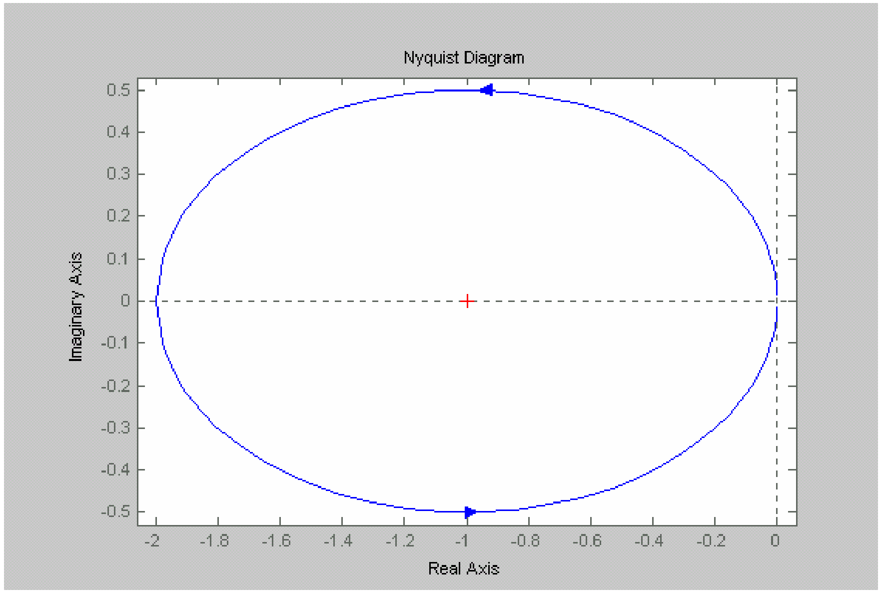
$$\operatorname{Im} G(j\omega) = -\frac{\omega}{1+\omega^2}$$

When $\omega = -\infty, G(j\omega) = 0.$

When $\omega < 0, \operatorname{Im} G(j\omega) > 0$

When $\omega \nearrow 0, \operatorname{Re} G(j\omega) \to -2$

$\omega > 0$ case is the mirror image



Closed loop is stable

$$\Leftrightarrow \quad -2 < -\frac{1}{k} < 0 \quad \Leftrightarrow \quad k > \frac{1}{2}$$

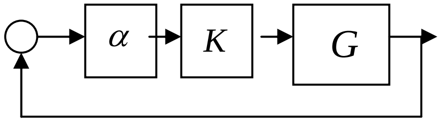
Check:

$$\text{num}(1 + kG(s)) = s^2 - 1 + k(s+2)$$

MATLAB command:

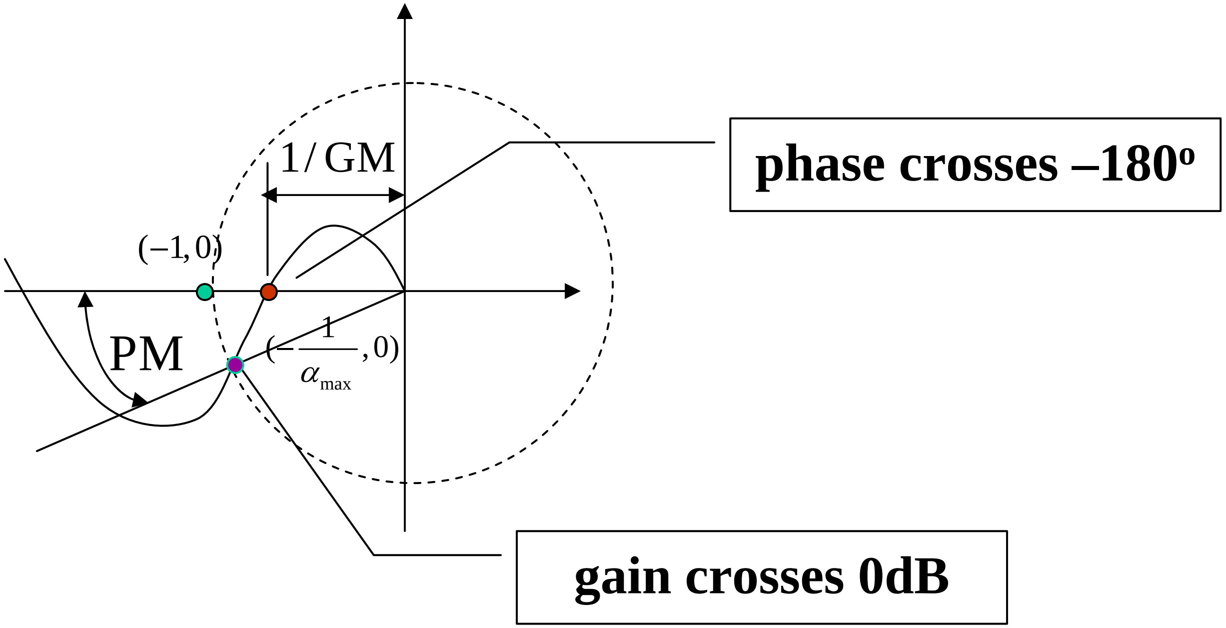
```
nyquist(G);
```

Gain/Phase Margins



Nominal value of α=1.

To evaluate the stability margin (how much α can vary), check the Nyquist plot of *GK:*

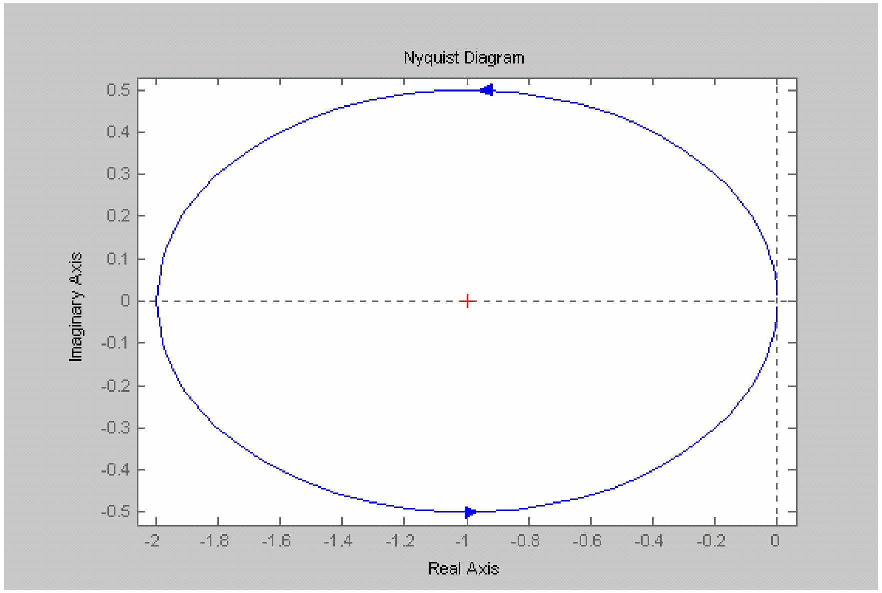


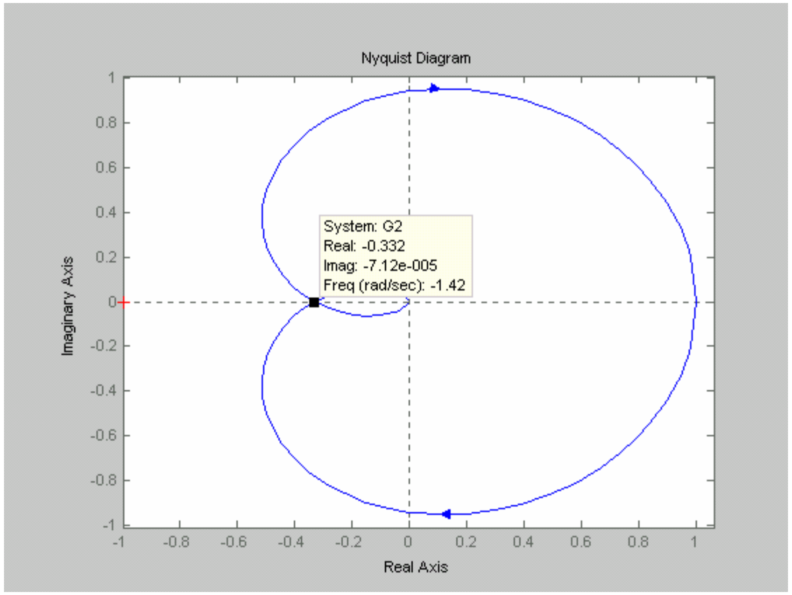
Example

$$G(s) = \frac{s+2}{s^2 - 1}$$

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

Nominal feedback gain *k*=1





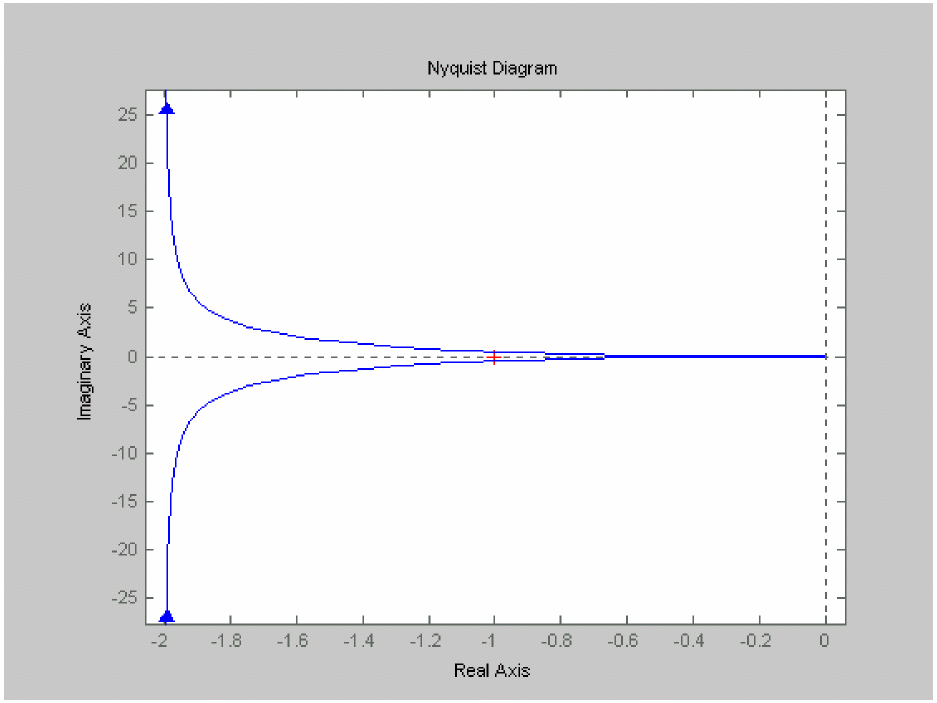
max gain: 1/2=0.5=-6dB GM

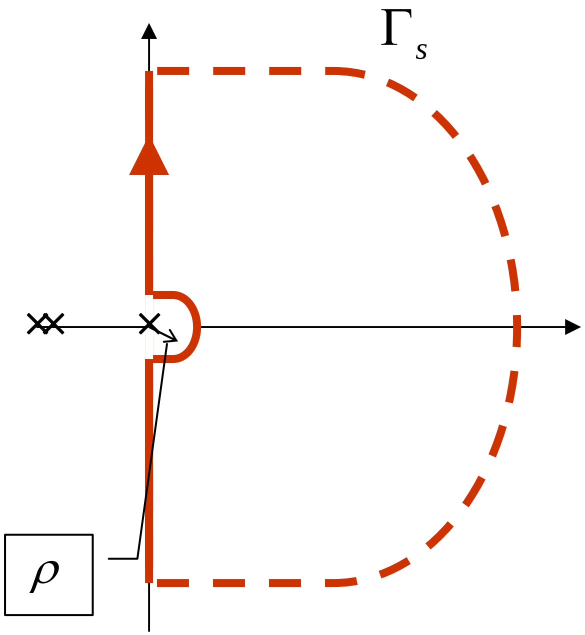
max gain: 1/.332=3.01=9.6dB GM

Zero on the imaginary axis

$$G(s) = \frac{1}{s(s+1)^2}$$

$G(j\omega) \to \infty$ as $\omega \to 0$ so Nyquist plot becomes unbounded.





$\rho \to 0$

which direction does the plot go at infinity?

$$G(s) = \frac{1}{s(s+1)^2} \approx \frac{1}{s} \text{ for } s \approx 0$$

$$G(\rho e^{j\phi}) \approx \rho^{-1} e^{-j\phi}$$

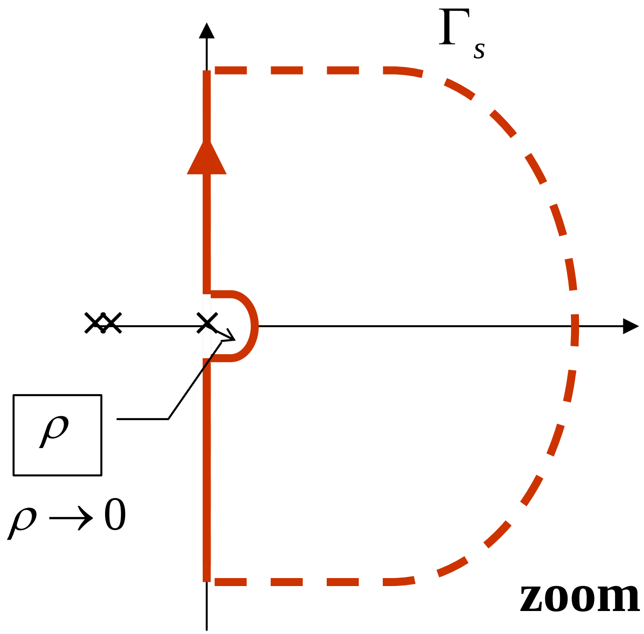
Zero on the imaginary axis

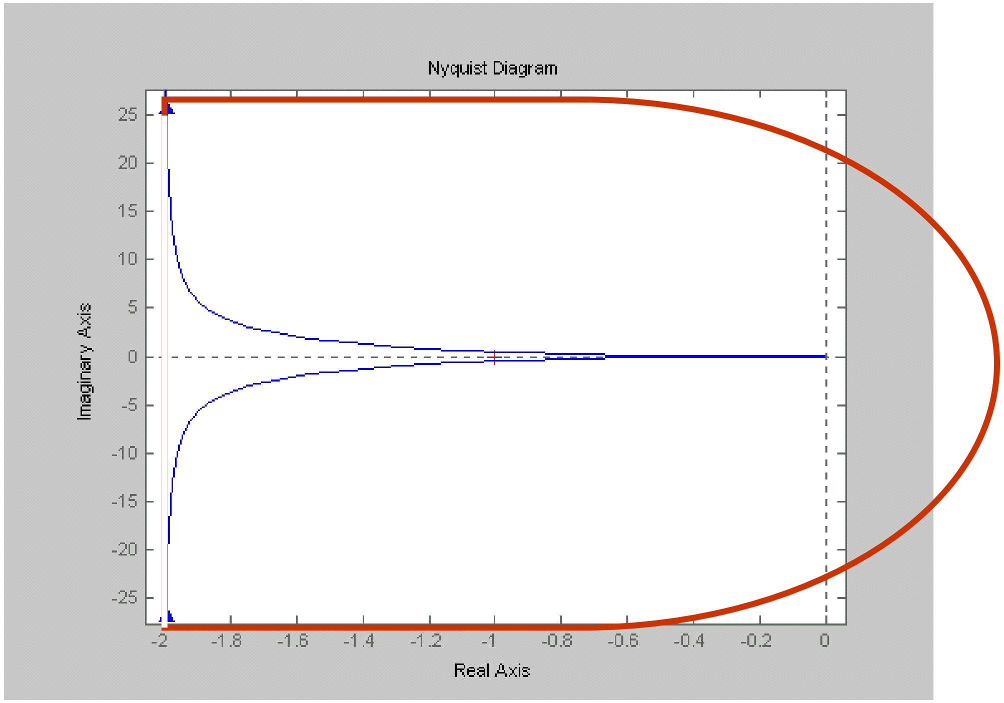
$$G(s) = \frac{1}{s(s+1)^2} \approx \frac{1}{s} \text{ for } s \approx 0$$

$$G(\rho e^{j\phi}) \approx \rho^{-1} e^{-j\phi}$$

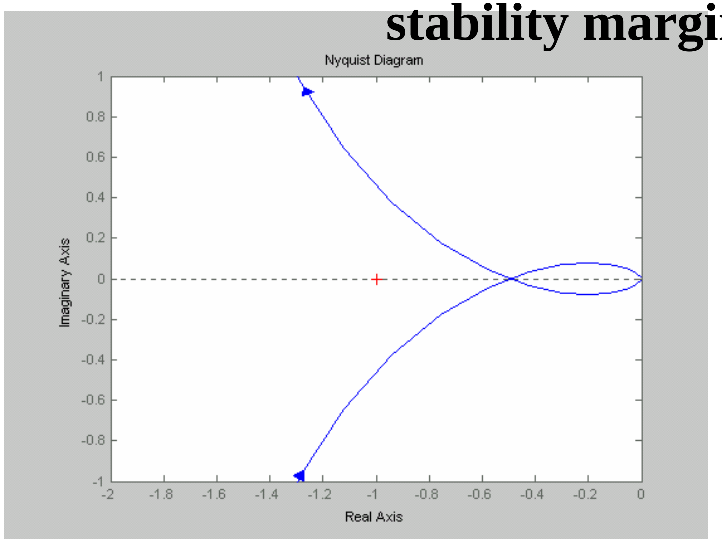
$$\phi : -\frac{\pi}{2} \to 0 \to \frac{\pi}{2}$$

$$e^{-j\phi} : j \to 1 \to -j$$





zoom in for stability margins



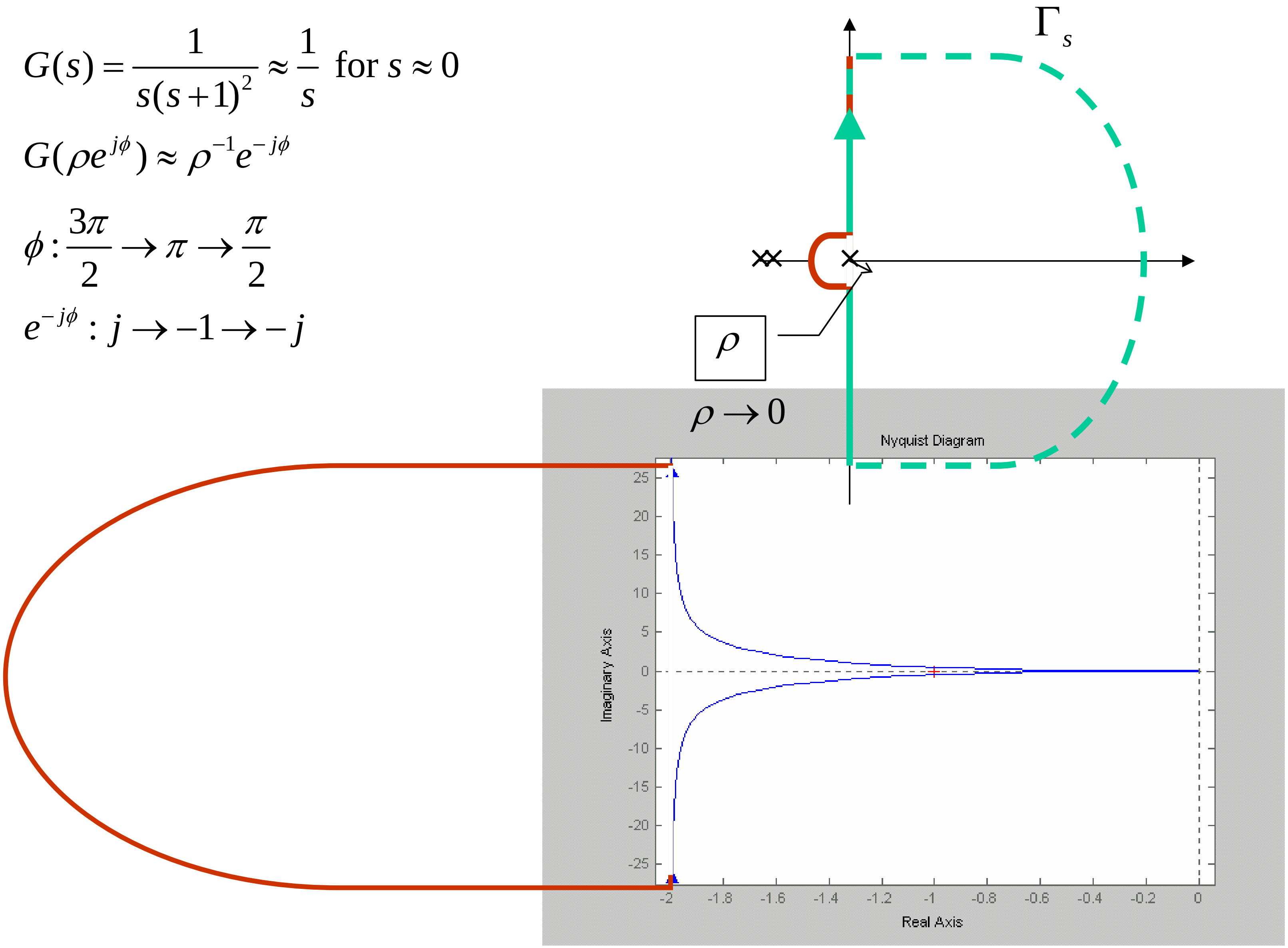
Zero on the imaginary axis

$$G(s) = \frac{1}{s(s+1)^2} \approx \frac{1}{s} \text{ for } s \approx 0$$

$$G(\rho e^{j\phi}) \approx \rho^{-1} e^{-j\phi}$$

$$\phi : \frac{3\pi}{2} \to \pi \to \frac{\pi}{2}$$

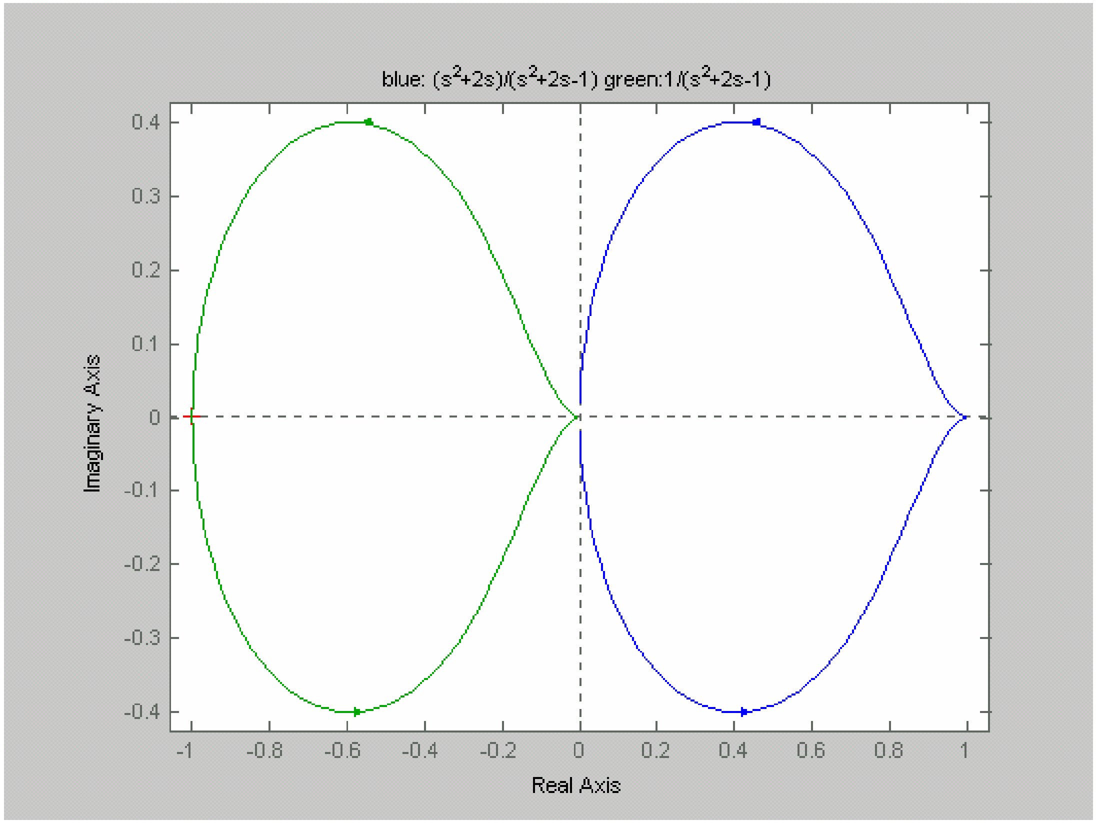
$$e^{-j\phi} : j \to -1 \to -j$$



Systems with Z=P

For strictly proper loop gain *L(s)* (more poles than zeros), we only need to evaluate the Nyquist plot for $s=j\omega$ since $L(s)\to 0$ as $\omega\to\pm\infty$. When *L(s)* has the same # of poles and zeros, this is no longer true. In this case, write *L(s)* as a constant + strictly proper transfer function. Plot the Nyquist plot of *L(s)* and shift the plot by the constant.

$$G(s)=\frac{s(s+2)}{(s^2+2s-1)}=1+\frac{1}{s^2+2s-1}$$

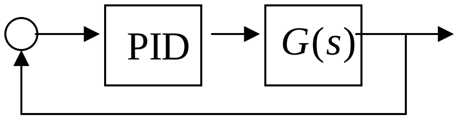


Exercise

Consider the lead filter $K_{lead}(s)=300(s+4)/(s+30)$ applied to the plant $G(s)=1/s(s+a)$:

- **Do the Nyquist plot of $G(s)$ by hand and compare with the MATLAB generated Nyquist plot.**
- **Use the MATLAB generated Nyquist plot for $G(s)K_{lead}(s)$ to evaluate the gain and phase margins. Compare with the results from using the Bode plot.**

Nyquist Plot Example



- **Consider the PID controller. Our goal is:**
  - **evaluate the range of k_i that achieves closed loop stability;**
  - **choose k_i and then evaluate gain/phase margin.**

$$G(s) = \frac{1}{s(s+a)} \qquad K(s) = k_p + k_d s + \frac{k_i}{s}$$

k_p, k_d are already chosen.

Nyquist Plot Example

- First step is the same as in root locus: isolate the gain of interest (k_i in this case). Write the return difference as:

$$1+G(s)K(s)=M(s)(1+k_iF(s))$$

$$1+G(s)K(s)=1+\left(\frac{1}{s(s+a)}\right)\left(k_p+k_ds+\frac{k_i}{s}\right)$$

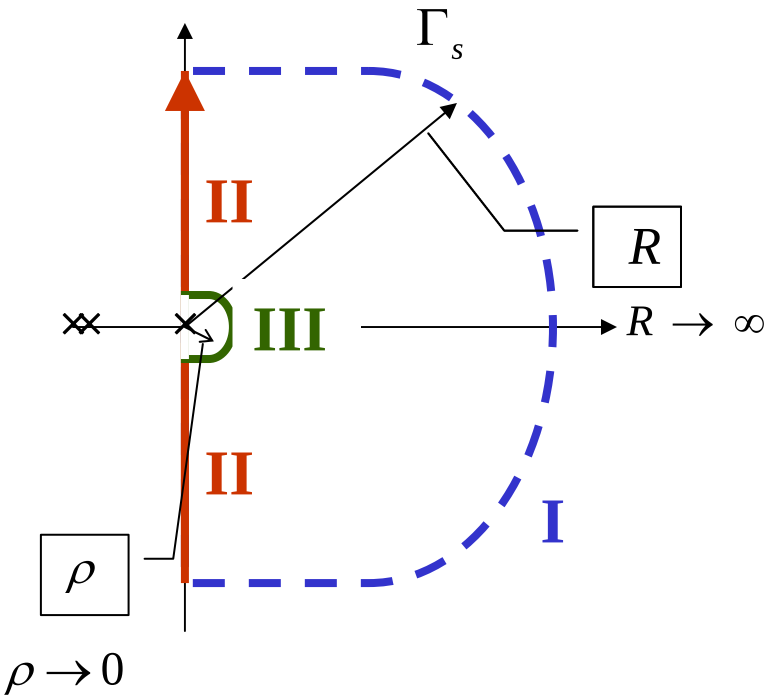
$$=1+\frac{k_p+k_ds}{s(s+a)}+\frac{k_i}{s^2(s+a)}$$

$$=\frac{s^2+(k_d+a)s+k_p}{s(s+a)}+\frac{k_i}{s^2(s+a)}$$

$$=\left(\frac{s^2+(k_d+a)s+k_p}{s(s+a)}\right)\left(1+k_i\underbrace{\frac{1}{s(s^2+(k_d+a)s+k_p)}}_{F(s)}\right)$$

Nyquist Plot Example

Now we do the Nyquist plot for $F(s) = \dfrac{1}{s(s^2 + (k_d + a)s + k_p)}$



I: *F(s)=0*

II: *s=jω:*

plot Re*F(jω)* vs. Im*F(jω)*

III: *s=ρ exp(jϕ)*

ϕ:-90º -> 0º -> 90º

Nyquist Plot Example

On II: $s = j\omega$

$$F(j\omega) = \frac{1}{-j\omega^3 - (k_d + a)\omega^2 + k_p j\omega}$$

$$= \frac{1}{-(k_d + a)\omega^2 + j\omega(k_p - \omega^2)}$$

$$= \frac{-(k_d + a)\omega^2 - j\omega(k_p - \omega^2)}{\left((k_d + a)\omega^2\right)^2 + \omega^2(k_p - \omega^2)^2}$$

$$= \frac{-(k_d + a)\omega^2 - j\omega(k_p - \omega^2)}{d(\omega)}$$

For $\omega \neq 0$ or ∞, $\operatorname{Re} F(j\omega) \neq 0$.

$\operatorname{Im} F(j\omega) = 0 \Leftrightarrow \omega^2 = k_p$

$$\operatorname{Re} F(j\omega) = -\frac{1}{(k_d + a)k_p}$$

At ω=-∞, $F(j\omega) = 0$.

As ω increases until $\omega = -\sqrt{k_p}$,

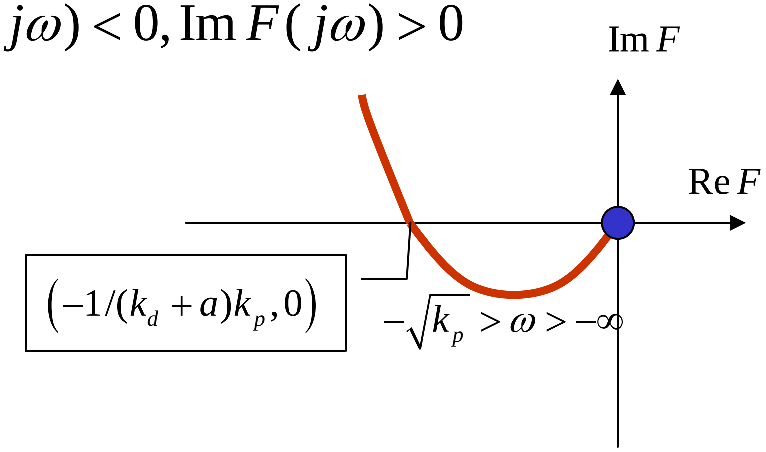
$\operatorname{Re} F(j\omega) < 0, \operatorname{Im} F(j\omega) < 0$

At $\omega = -\sqrt{k_p}$,

$$\operatorname{Re} F(j\omega) = \frac{1}{(k_d + a)k_p}, \operatorname{Im} F(j\omega) = 0$$

For $0_- > \omega > -\sqrt{k_p}$,

$\operatorname{Re} F(j\omega) < 0, \operatorname{Im} F(j\omega) > 0$



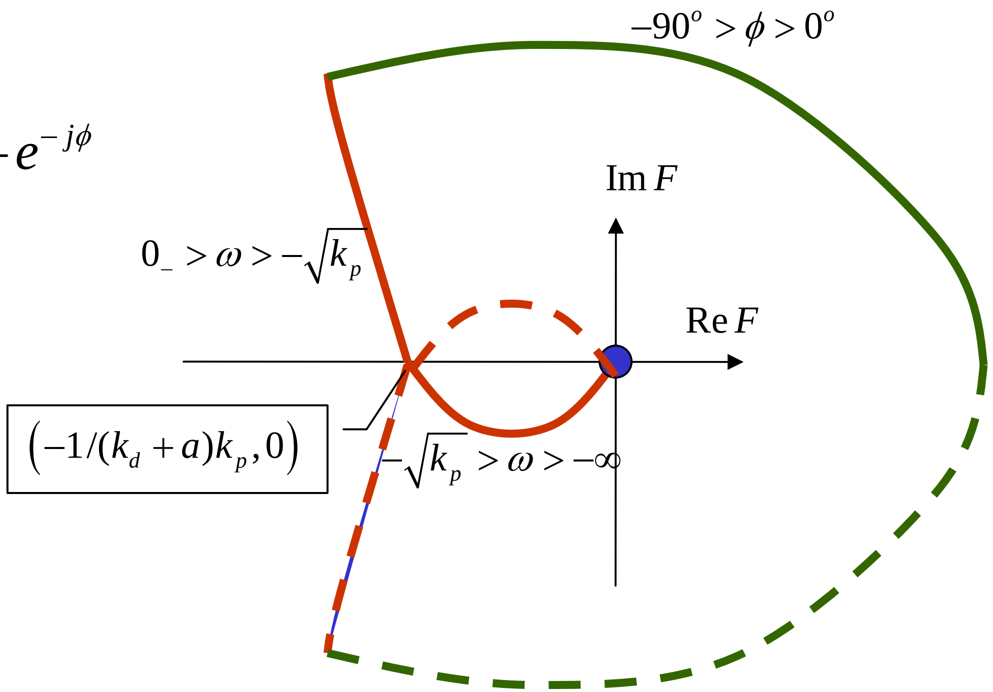
Nyquist Plot Example

On III: $s = \rho e^{j\phi}$

Since $\rho \to 0, F(s) \approx \dfrac{1}{sk_p} = \dfrac{1}{\rho k_p} e^{-j\phi}$

As $\phi : -90^o \to 0^o \to 90^o$,

$\measuredangle F(s) : 90^o \to 0^o \to -90^o$



Range of k_i for stability:

$$-\infty < -\frac{1}{k_i} < -\frac{1}{(k_d + a)k_p}$$

or

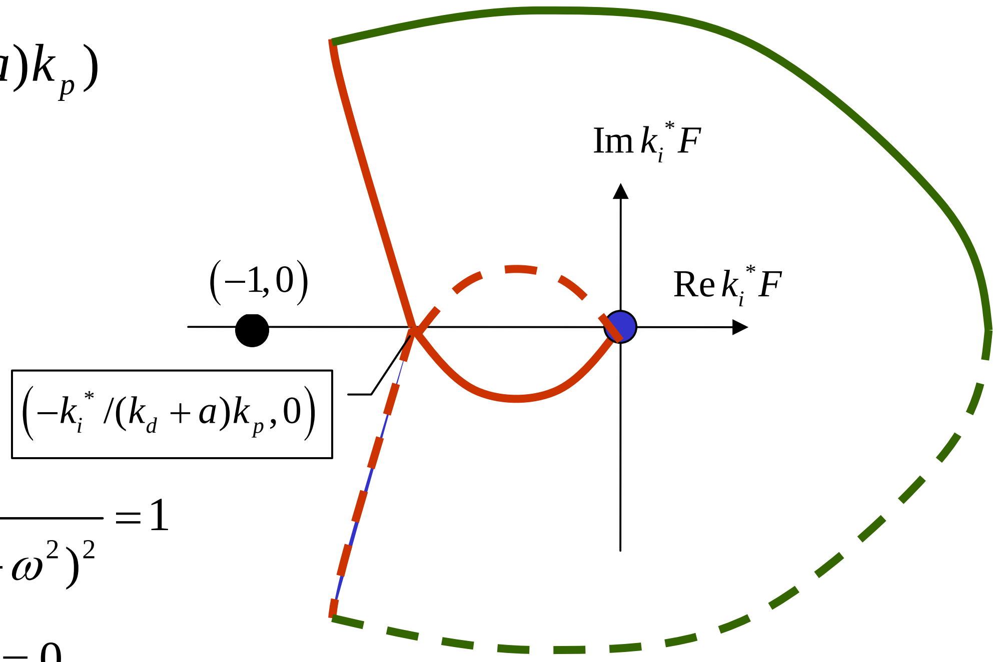
$$0 < k_i < (k_d + a)k_p$$

Nyquist Plot Example

If k_i is chosen to be k_i^*, the Nyquist plot is simply scaled by k_i^*.

Gain Margin: $20\log_{10}(k_i^* / (k_d + a)k_p)$

Phase crossover frequency: $\sqrt{k_p}$



To find the Phase Margin,

solve ω from $\left|k_i^* F(j\omega)\right| = 1$.

$$\left|k_i^* F(j\omega)\right|^2 = \frac{k_i^{*2}}{\left((k_d + a)\omega^2\right)^2 + \omega^2 (k_p - \omega^2)^2} = 1$$

$$\omega^6 + ((k_d + a)^2 - 2k_p)\omega^4 + k_p^2 \omega^2 - k_i^{*2} = 0$$

The only positive real root is the gain crossover frequency ω_{cg},

$$\text{PM} = 180^\circ + \measuredangle F(j\omega_{cg})$$

Bode Gain/Phase Relationship

- **Ideally, we would like to adjust gain and phase of the loop gain independently to achieve desired bandwidth and gain/phase margins. However Bode showed in 40's that there is a fixed relationship between them:**

$$\angle G(j\omega_o) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{dM}{du}\right)W(u)du$$

$$M = \ln\left|G(j\omega)\right|$$

$$u = \ln(\omega/\omega_o)$$

$$W(u) = \ln(\coth\left|u\right|/2) \approx \frac{\pi^2}{2}\delta(u)$$

Bode Gain/Phase Relationship

- Suppose we would like to have
  –20 *n* dB/decade roll-off in gain plot, then *dM/du* ≈ *-n* and $\angle G(j\omega_o) \approx -n\,\pi/2$.

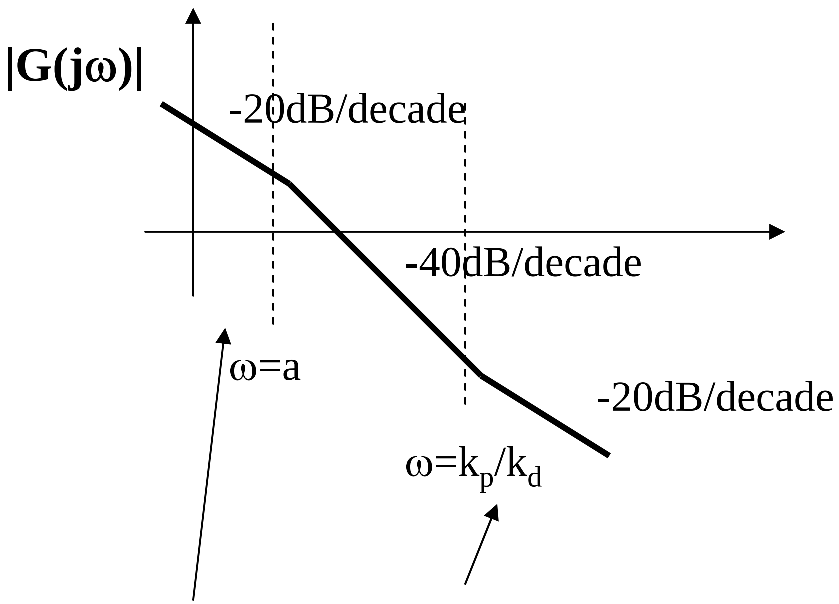
$$\angle G(j\omega_o) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{dM}{du}\right)W(u)du$$

$$W(u) = \ln(\coth|u|/2) \approx \frac{\pi^2}{2}\delta(u)$$

- Implication to control design: Around 0dB of *KG* (1 decade centered around the gain crossover), the gain roll-off should be around –20dB/decade to ensure approximately 90º phase margin.

Example

- Consider PD control of $G(s)=1/(s^2+as)$. Loop Gain $L(s)= (k_p+k_d s)/(s^2+as)$.

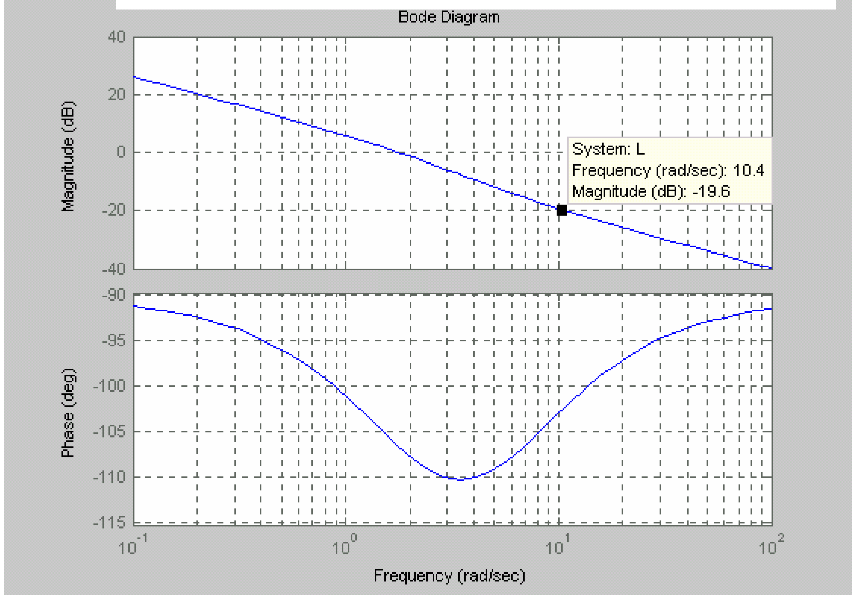


Gain crossover should be chosen around here for good phase margin.

Example

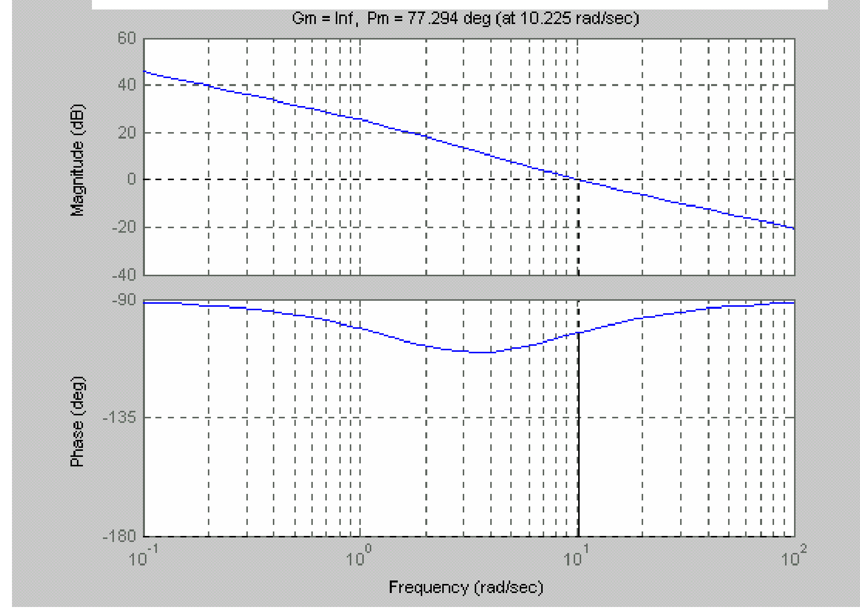
- **Suppose desired bandwidth = 10rad/sec. Choose k_p/k_d = 5 rad/sec and adjust k_d so the gain crossover is at 10rad/sec.**

Bode plot of $(s+5)/(s^2+as)$



Add 19dB ➔ k_d= 8.9

Bode plot of 8.9 $(s+5)/(s^2+as)$



gain crossover at 10.2rad/s
PM = 77.3 deg