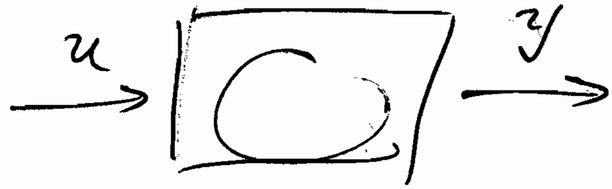
$+$ $\times$



$$\begin{cases} \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u \\ y(0),\ \dot{y}(0) \text{ given} \end{cases}$$

$$Y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} U(s)$$

transfer function

$$\dot{x} = Ax + Bu \quad ; \quad x(0) = x_0$$

$$Y = Cx + Du$$

<u>Controller canonical form</u>

$$Y(s) = \frac{n(s)}{d(s)} U(s) \rightarrow \xi(s)$$

$$Y(s) = \cancel{n(s)\xi(s)} \qquad \xi(s) = \frac{U(s)}{d(s)}$$

$$n(s)\, \xi(s)$$

$$Y(s) = (b_1 s + b_0)\, \xi(s) \qquad \xi(s) = \frac{U(s)}{s^2 + a_1 s + a_0}$$

$$\boxed{\ddot{\xi} + \underline{\underline{a_1}}\,\dot{\xi} + \underline{\underline{a_0}}\,\xi = u}$$

$x_1 = \xi \qquad \dot{x}_1 = x_2$

$x_2 = \dot{\xi} \qquad \dot{x}_2 = -a_0 x_1 - a_1 x_2 + u$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$\ddddot{\xi} + 3\dddot{\xi} - 2\ddot{\xi} + 6\dot{\xi} + 5\xi = u$$

$x_1 = \xi \qquad \dot{x}_1 = x_2$

$x_2 = \dot{\xi} \qquad \dot{x}_2 = x_3$

$x_3 = \ddot{\xi} \qquad \dot{x}_3 = x_4$

$x_4 = \dddot{\xi} \qquad \dot{x}_4 = -5x_1 - 6x_2 + 2x_3 - 3x_4 + u$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -6 & 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\left.\begin{aligned} x_1 &= \xi + \dot{\xi} \\ x_2 &= \xi - \dot{\xi} \\ x_3 &= \ddot{\xi} + \dddot{\xi} \\ x_4 &= \ddot{\xi} - \dddot{\xi} \end{aligned}\right\} \rightarrow \text{different } (A, B)$$

$$y = b_1 \dot{\xi} + b_0 \xi = b_0 x_1 + b_1 x_2$$

$$= \underbrace{[b_0 \quad b_1]}_{C} x + \underbrace{0}_{D} \cdot u$$

Given (A, B, C, D)

$$\dot{x} = Ax + Bu \qquad \underline{x \in \mathbb{R}^n} \quad \text{n dimensional real vector}$$

$$y = Cx + Du$$

$$s \underbrace{X(s)} - x(0) = A X(s) + B U(s)$$

$$Y(s) = C X(s) + D U(s)$$

n×n identity matrix

$$\underbrace{(sI - A)} X(s) = x(0) + B U(s)$$

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} B U(s)$$

$$Y(s) = \underbrace{C (sI - A)^{-1} x(0)}_{I.C.} + C (sI - A)^{-1} B U(s) + D U(s)$$

transfer function: $C (sI - A)^{-1} B + D$

$$C(sI-A)^{-1}B+D = \frac{n(s)}{d(s)}$$

$$(sI-A)^{-1} = \frac{\mathrm{Adj}\,(sI-A)}{\det(sI-A)}$$

$n \times n$ polynomial matrix

Scalar polynomial

$$\underset{\sim}{\overset{1\times n}{C}}\ \underbrace{\mathrm{Adj}(sI-A)}\ \underset{\sim}{\overset{n\times 1}{B}} + D\,\det(sI-A) = n(s)$$

$$\det(sI-A) = d(s)$$

poles: $\det(sI-A)=0$

$\Longleftrightarrow \quad (sI-A)x = 0$

$\Longleftrightarrow \quad Ax = sx$

$$y \leftarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \leftarrow u$$

$$\underbrace{y \leftarrow \boxed{M^{-1}}}_{y'} \leftarrow \begin{bmatrix} -.37 & 0 \\ 0 & 5.37 \end{bmatrix} \leftarrow \underbrace{\boxed{M}}_{u'} \leftarrow u$$

$$\dot{x} = A x$$

$$M = \begin{bmatrix} | & | & | & | & | \end{bmatrix} \text{ eigenvectors}$$

$$M^{-1}\dot{x} = \underbrace{M^{-1} A M}\, M^{-1} x$$

$$z \triangleq M^{-1} x \qquad x = M z$$

$$\dot{z} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} z$$

$$Y = Cx = C M (\overbrace{M^{-1} x}^{z})$$

$$A[\underbrace{x_1 | \cdots | x_n}_{M}] \qquad \underline{\underline{Ax_i = \lambda x_i}}$$

$$= [Ax_1 | \cdots | Ax_n]$$

$$= [\lambda_1 x_1 | \cdots | \lambda_n x_n]$$

$$= [\underbrace{x_1 | \cdots | x_n}_{M}] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

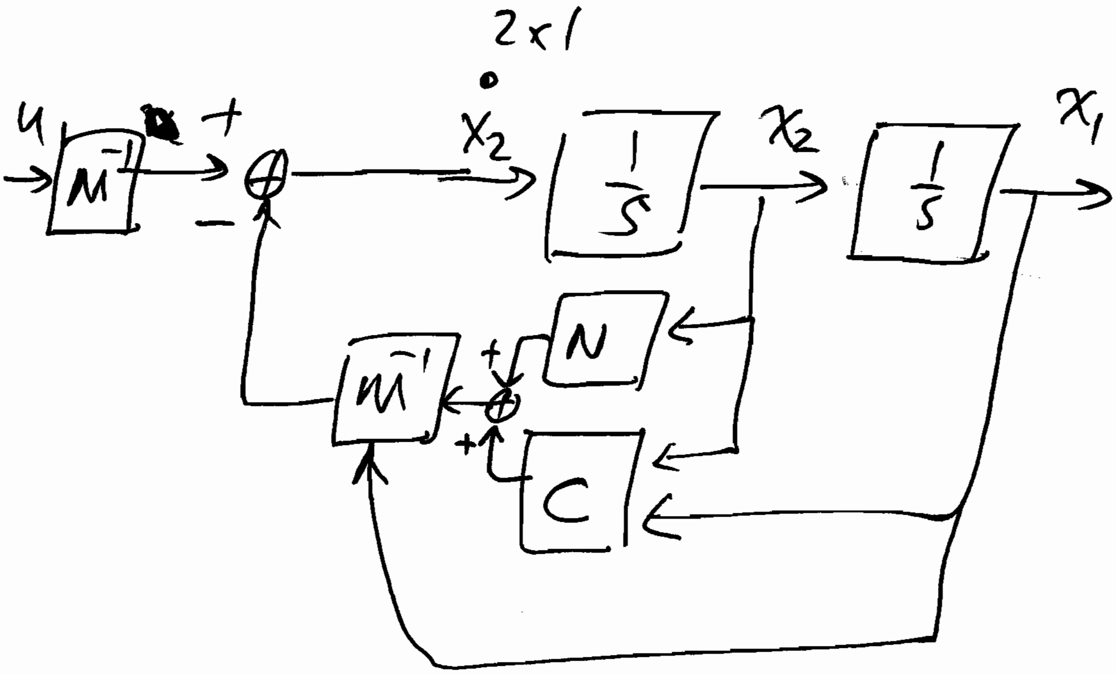
$$x_1 = q$$

$$x_2 = \dot{q}$$

$$\dot{x}_1 = \dot{q} = x_2$$

$$\dot{x}_2 = \ddot{q} = M^{-1}(-C - N + u)$$

$$\underbrace{\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{x}}}_{4 \times 1} = \underbrace{\begin{bmatrix} x_2 \\ -M^{-1}(x_1)\left(C(x_1, x_2)x_2 + N(x_2)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} u}$$



Project: Linearize about $(q_d, \underset{\dot q}{0})$

4 states

$$M(q_d)\ddot q + D\dot q = \overset{2\times 1}{u}$$

$\mathrm{sgn}(\dot\theta_i)$

$$y = q \leftarrow 2\times 1$$

$x_1 = q$

$x_2 = \dot q$

$x_1 = q$

$x_2 = M(q_d)\dot q$

$\dot x_1 = x_2$

$\dot x_2 = -M^{-1}(q_d) D x_2 + M^{-1}(q_d) u$

$\dot x_1 = M^{-1}(q_d) x_2$

$\dot x_2 = M(q_d)\ddot q$

$= -D M^{-1}(q_d) x_2 + u$

$$\dot x = \underbrace{\begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ 0_{2\times 2} & -M^{-1}(q_d) D \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0_{2\times 2} \\ M^{-1}(q_d) \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} I_{2\times 2} & 0_{2\times 2} \end{bmatrix}}_{C} x + \underbrace{0_{2\times 2}}_{D} u$$

$$\bar{Y}(s) = \underbrace{\overbrace{C\,(sI-A)^{-1}B}^{G(s)}}_{2\times 2}\,\bar{U}(s)$$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$\to G_{11}(s)$

$\to G_{22}(s)$

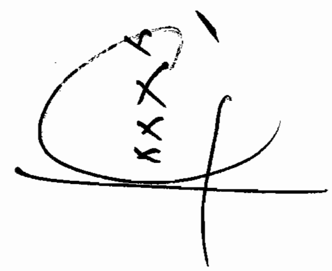
$$\dot{x} = A x + \boxed{B u}$$

← direct control design in the state space

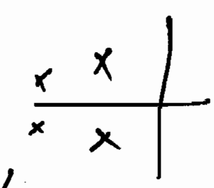
$$\underline{u = -Fx}$$

← full state feedback

$$\dot{x} = \underbrace{(A - BF)}\,x$$



If (A, B) is "controllable", then



the eigenvalues of $(A - BF)$ may be arbitrarily assigned by using $\mathbf{F}$.

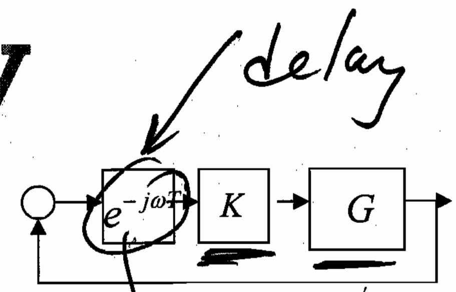
→ modulo complex conjugate pairs

Controllability of (A,B) $\quad \operatorname{rank}(\operatorname{ctrb}(A,B)) = n$

$$\operatorname{rank}\underbrace{\left[B \mid AB \mid \cdots \mid A^{n-1}B\right]} = n$$

Controllability matrix

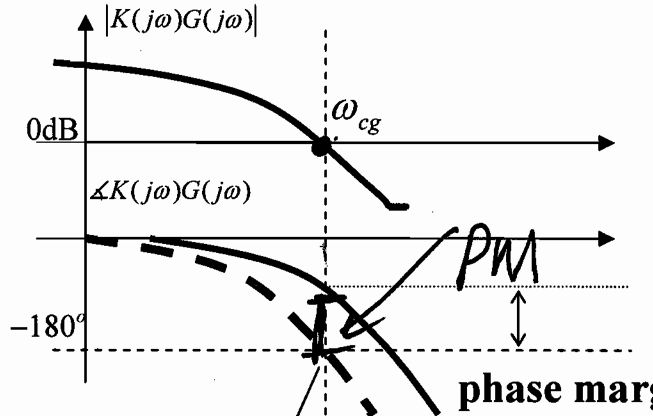
Time Delay



Time delay adds a phase shift of $-\omega T$.

Boundary of stability: $\mathrm{PM} = \omega_{cg} T$

Maximum delay: $T_{max} = \dfrac{\mathrm{PM}}{\omega_{cg}}$



Additional phase lag: $\omega_{cg} T$

Phase plot with added phase lag from time delay.

$$\measuredangle e^{-j\omega T} = -\omega T$$

$$\omega_{cg} T = PM \quad \therefore T = \frac{PM}{\omega_{cg}}$$

Properties

- poles of $G(s)$ = eigenvalues of A
- (A,B,C,D) is not unique for a given $G(s)$
- MATLAB tools:

```
G=ss(A,B,C,D); [A,B,C,D]=ssdata(G);
```

```
Gtf= ss2tf(G);     G = tf2ss(Gtf);
Gzp= ss2zp(G);     G = zp2ss(Gzp);
```

State Space Description

$$G(s)=\frac{b_1 s+b_0}{s^2+a_1 s+a_0}$$

$$x_1 = x,\ x_2 = \dot{x}$$

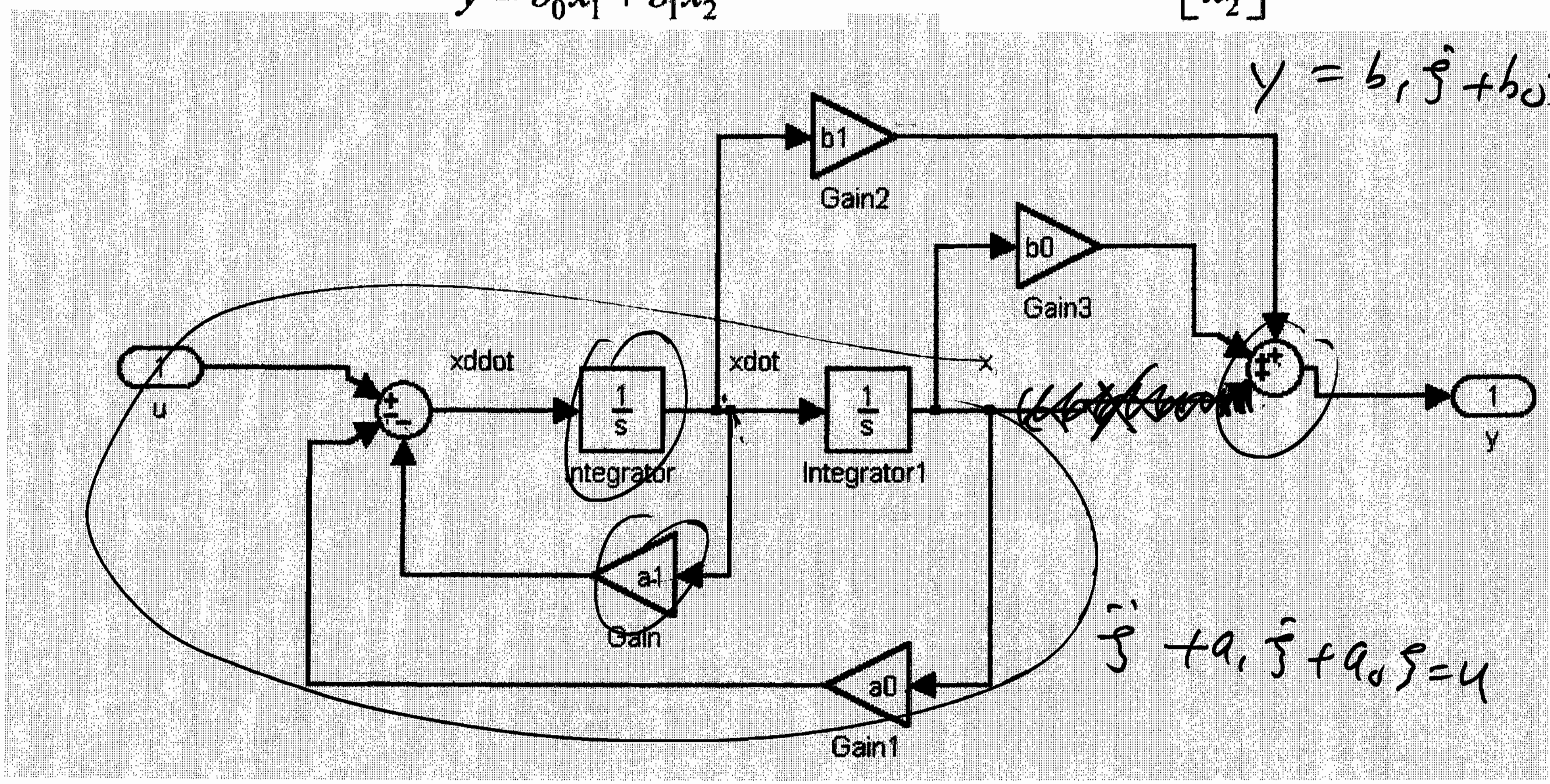
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a_0 x_1 - a_1 x_2 + u$$

$$y = b_0 x_1 + b_1 x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}b_0 & b_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0\cdot u$$



Pole Placement

How do we choose *F* to place the closed loop poles?

- **Consider (*A*,*B*) in controllable canonical form:**

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Companion matrix

- **Coefficients of det(sI-(*A*-*BF*)) (closed loop characteristic polynomial) are given by**

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

$$A - BF = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 - f_1 & -a_1 - f_2 & \cdots & -a_{n-1} - f_n \end{bmatrix}$$

Homework (due 11/29/05)

- Put the 2-link system in the project in state space form.
- Linearize the system. Find poles of the system.
- Obtain $\tau_1 \rightarrow \theta_1$ and $\tau_2 \rightarrow \theta_2$ transfer functions.
- Consider the following the design spec: closed loop stability, rise time about ½ sec, 2% settling time, less than 5% overshoot, no more than 1 sec, 6dB gain margin (gain scaling from ½ to 2) and 10ms delay tolerance.

$\theta_1 \to \tau_1$
$\theta_2 \to \tau_2$

  - Use root locus to design a controller to achieve the design spec.
  - Assume full state is available (e.g., by using the washout filter). Use state space pole place design method to achieve the design spec.
- Simulate the linear and nonlinear system responses using the above controllers.

Project

Project spec:

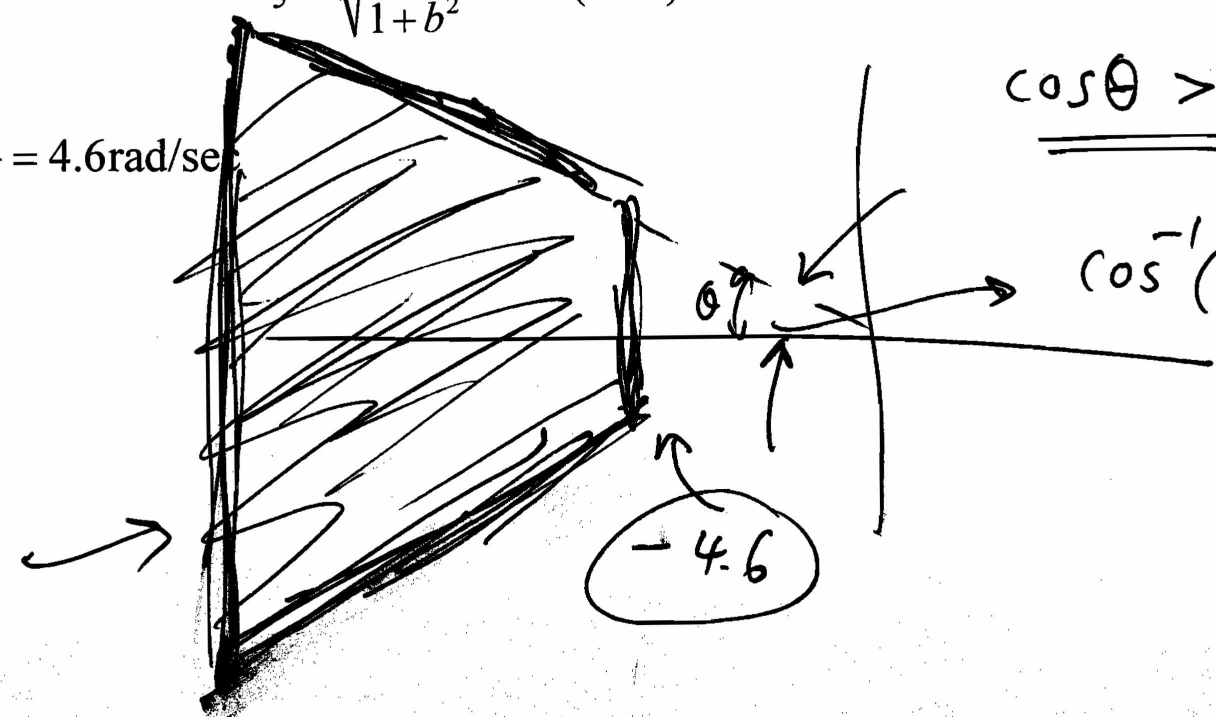
$t_r < .5\,\text{sec},\ t_s < 1\,\text{sec},\ M_p < 0.05$

Rule of thumb:

$\omega_n \geq \dfrac{1.8}{t_r} = 3.6\ \text{rad/sec}$ $\quad \zeta \geq \zeta(M_p) \quad \zeta / \sqrt{1-\zeta^2} = \ln M_p / (-\pi) = b$

$$\zeta = \sqrt{\frac{b^2}{1+b^2}} = .7 \ \ (70\%)$$

$\sigma = \zeta\omega \geq \dfrac{4.6}{t_s} = 4.6\text{rad/sec}$



$\cos\theta > .7$

$\cos^{-1}(.7)$