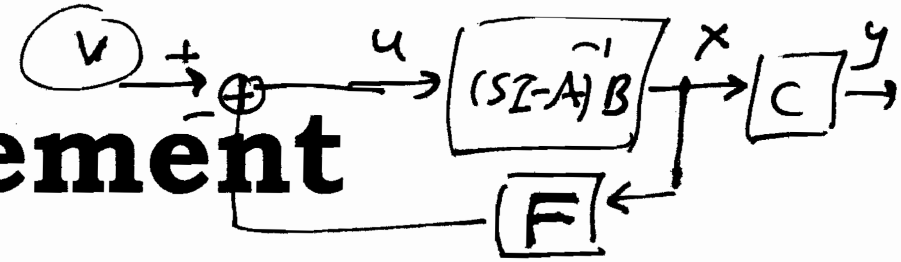


Pole Placement

- **Choose *F* based on the desired closed loop characteristic polynomials.**

$$f_1 = a_{cl_0} - a_0, f_2 = a_{cl_1} - a_1, \cdots, f_n = a_{cl_{n-1}} - a_{n-1}$$

eig(A) = open loop poles

$$\boxed{\dot{x} = Ax + Bu}$$

$$\cancel{y = Cx}$$

$\dot{x} = Ax + Bu$

$= (A - BF)x + Bv$

$y = Cx$

Example: $G(s) = 1/(s^2 + 2.427s)$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2.427 \end{bmatrix} B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Desired poles at (-10,-15)

Desired closed loop characteristic poly:

$s^2 + 25s + 150$

$F = [150 \quad 25 - 2.427] = [150 \quad 22.573]$

Suppose x is available
(full state feedback)

Linear full state feedback

- **MATLAB tools: `place`, `acker`**

$u = -Fx$

$\underbrace{\text{Closed loop eigenvalues}}$

$\dot{x} = (A - BF)x$

Discrete Time Systems

Under zero-order-hold, equivalent discrete time system is

Sampler

$$u_k \xrightarrow{} \underset{\text{ZOH}}{\boxed{D/A}} \xrightarrow{u} \underbrace{\boxed{(sI-A)^{-1}B}} \xrightarrow{x} \boxed{A/D} \xrightarrow{} x_k$$

$$x_{k+1} = A_d x_k + B_d u_k$$

$x_k = x(k t_s)$

t_s = Sampling period

Full state feedback control:

$$u_k = -F(x_k - x_d)$$

Pole placement can again be used to place the closed loop poles ***eig(*** $A_d - B_d F$ ***)*** **within the unit circle.**

Full State Observer

$$\dot{\hat{x}} = \underbrace{A\hat{x} + B\cancel{u}}_{\text{replica of plant}} \underbrace{- L(\overbrace{\underbrace{C\hat{x} + D\cancel{u}}_{\text{estimated output}} - \overset{\substack{Cx+D\cancel{u} \\ \shortparallel}}{y}}^{})}_{\substack{\text{output estimation error} \\ \text{correction based on} \\ \text{output estimation error}}} \qquad \hat{x}(0) = 0$$

$$\dot{x} = Ax + \cancel{B}u \qquad x(0) = x_0$$

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + L(-C(x - \hat{x}))$$

$$e_x \overset{\Delta}{=} x - \hat{x} \qquad x(0) - \hat{x}(0) = x_0$$

$$\dot{e}_x = (A - LC)e_x \qquad e_x(0) = x_0$$

How do we choose *L*?

Fact: $eig(A)=eig(A^T)$

$$(A-LC)^T = A^T - (LC)^T$$

Therefore, $eig(A-LC) = eig(A^T-C^TL^T)$

$$= A^T - C^T L^T$$

This is exactly the same as the full state feedback case except (A,B) is replaced by (A^T,C^T).

Rule of Thumb: Observer poles should be 5-10 times faster than the controller poles.

$$\dot{x} = Ax + Bu(t); \quad x(0) = x_0$$

Zero input solution:

Scalar

$$\dot{x} = ax \; ; \; x(0) = x_0$$

$$x = e^{at} x_0$$

$$= \sum_{i=0}^{\infty} \frac{(at)^i}{i!}$$

$$\dot{x} = Ax \; ; \; x(0) = x_0$$

$$x = \boxed{e^{At}} x_0 \qquad \dot{x} = A \underbrace{e^{At} x_0}_{x}$$

$$e^{At} \triangleq \boxed{\sum_{i=0}^{\infty} \frac{(At)^i}{i!}}$$

Verify: $\boxed{\frac{d}{dt} e^{At} = A e^{At}}$

$$e^{At}\Big|_{t=0} = I$$

Zero state solution

$$\dot{x} = Ax + B\,\delta(t)$$

$$\dot{x} - Ax = B\,\delta(t)$$

$$e^{-At}(\dot{x} - Ax)$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) \qquad = e^{-At}B\,\delta(t)$$

$$= -\underbrace{A\,e^{-At}}_{\text{commute}}\,x(t) + e^{-At}\dot{x}(t)$$

$$= e^{-At}\left(\dot{x}(t) - Ax(t)\right)$$

$$\therefore \frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}B\,\delta(t)$$

Integrate both sides:

$$e^{-At}x(t) = \int_0^t e^{-A\tau}B\,\delta(\tau)\,d\tau$$

$$= B$$

$$\therefore x(t) = e^{At}B \longleftarrow \text{impulse response}$$

Complete response of

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

$$x(t) = \underbrace{e^{At} x_0}_{\text{zero input}} + \underbrace{\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau}_{\substack{\text{convolution between} \\ \text{zero state response \& input}}}$$

$$x((k+1)t_s) = e^{At_s} x(kt_s)$$

$$+ \int_0^{t_s} e^{A(t_s-\tau)} B\, u_k\, d\tau$$

$$= \overbrace{e^{At_s}}^{A_d} x(kt_s)$$

$x_{k+1} = a x_k$

$x_0, a x_0, a^2 x_0, a^3 x_0$

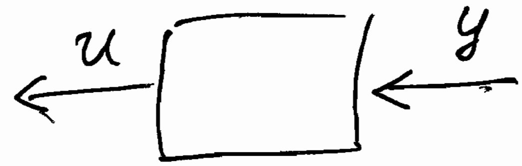
$$+ \underbrace{\left[\int_0^{t_s} e^{A(t_s-\tau)} B\, d\tau\right]}_{B_d} u_k$$

$$x_{k+1} = A_d x_k + B_d u_k = (A_d - B_d F)\, x_k$$

eigenvalues

$$u_k = -F x_k$$

Controller



$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$

$$u = -F\hat{x}$$

$$\boxed{\begin{aligned} \dot{\hat{x}} &= (A - BF - LC)\hat{x} + Ly \\ u &= -F\hat{x} \end{aligned}}$$