Robotics & Automation

Lecture 02
Rigid Body Orientation

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Transformation of vectors: \( v_a = R_{ab} v_b \)

Transformation of linear transforms: \( L_a = R_{ab} L_b R_{ba} \)

\( R \in SO(3) \) has the interpretation of representation of one frame in another, direction cosine matrix, coordinate transformation matrix, and rotation of frames.

Any rotation matrix \( R \) can be written as (Euler-Rodrigues Formula):

\[
R = I_3 + \sin \theta \hat{k} + (1 - \cos \theta) \hat{k}^2.
\]
Consider $e^{\hat{k}\theta}$. Eigenvalues of $e^{\hat{k}\theta}$ are $1, e^{\pm j\theta}$ (since $\hat{k}$ has eigenvalues $\{0, \pm j\}$).

From Cayley-Hamilton Theorem, we have

$$e^{\hat{k}\theta} = a_0 I + a_1 \hat{k} + a_2 \hat{k}^2.$$ 

Substituting the eigenvalues for $\hat{k}$, we have

$$1 = a_0 \quad e^{j\theta} = a_0 + a_1 j - a_2 \quad e^{-j\theta} = a_0 - a_1 j - a_2.$$ 

Therefore, $a_0 = 1$, $a_1 = \sin \theta$, $a_2 = 1 - \cos \theta$, which results in the same expression as the Euler-Rodrigues formula. Also, the eigenvalues of $e^{\hat{k}\theta}$ are $\{1, e^{\pm j\theta}\}$ with $k$ as the eigenvector corresponding to the eigenvalue 1.

Euler’s (Rotation) Theorem: Given $R \in SO(3)$, there exists $k$ ($k \in \mathbb{R}^3$ and $\|k\| = 1$) and $\theta \in \mathbb{R}$ such that $R = e^{\hat{k}\theta}$.

In words, every rotation corresponds to a single rotation about a unit vector. $k$ is called the equivalent axis and $\theta$ the equivalent angle.
Given \((k, \theta)\), we can find a corresponding \(R \in SO(3)\). How do we find \((k, \theta)\) for a given \(R \in SO(3)\)? If so (i.e., Euler’s theorem), is it unique?

Given \(R \in SO(3)\), find \((k, \theta)\). Exercise: show that

\[
\sin \theta = \pm \frac{\|R - R^T\|}{2}, \quad \cos \theta = \frac{\text{tr}R - 1}{2}.
\]

Therefore, \(\theta\) may be recovered from \(\theta = \text{atan2}(\sin \theta, \cos \theta)\) (with sign ambiguity). Equivalent axis, \(k\), may be solved from (if \(\sin \theta > 0\)):

\[
k = \frac{(R - R^T)^\vee}{2 \sin \theta}.
\]

What if \(R = R^T = I\)?
**Example**

Let’s consider a rotation about the $z$ axis, then

$$ R = I + \sin \theta \hat{z} + (1 - \cos \theta) \hat{z} \hat{z}. $$

Verify it is the same as before.

In general, use MATLAB `expm` function to verify exponential formula
Other Representations: Unit Quaternion

Unit Quaternion (Euler Parameters) \((q_0, q_v)\) (scalar and vector quaternion) with \(q_0^2 + q_v^T q_v = 1\).

\[
R(q) = I + 2q_0\hat{q}_v + 2\hat{q}_v\hat{q}_v, \quad q_0 = \cos(\theta/2), q_v = \sin(\theta/2)k.
\]

Can you write \((q_0, q_v)\) in terms of \(R\) directly (without computing \((k, \theta)\) first)?

Quaternion is a generalization of complex number: \(q_0 + q_1i + q_2j + q_3k\) with \(i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\). Quaternion multiplication, \(qaqb\), is given by

\[
(q_{a,0}I_4 + \begin{bmatrix}
0 & -q_{{a,v}}^T \\
q_{{a,v}} & \hat{q}_{{a,v}}
\end{bmatrix}) \begin{bmatrix}
qb,0 \\
qb,v
\end{bmatrix}.
\]

Nice property of quaternion: \(R(q_a)R(q_b) = R(q_aq_b)\). Also, we shall see that quaternion is a nonsingular representation.
Vector Quaternion and Gibbs Vector

Vector Quaternion: \( q_v = \sin(\theta/2)k \) (replace \( q_0 \) by \( +\sqrt{1 - \|q_v\|^2} \)).

Gibbs Vector or Euler-Rodrigues parameters: \( s = \tan(\theta/2)k \). (Write \( R \) in terms of \( s \).)

Note that in contrast to Euler angles, \( R \) is either a polynomial or rational functions of the representation parameters – no trigonometric functions!
Other Representations: Euler Angles

Euler angles: three consecutive principal axes rotations, e.g., 313 Euler angles:

\[ R_{03} = R_{01} (\beta_1) R_{12} (\beta_2) R_{23} (\beta_3). \]

where \( R_{i-1,i} (\beta_i) = e^{a_{i-1} \beta_i}, a_{i-1} \) is one of the coordinate vectors of the \( i - 1 \)th frame.
Euler Angle (Cont.)

\[ R_{03} = e^{\hat{z}\beta_1} e^{\hat{x}\beta_2} e^{\hat{z}\beta_3} \]

\[ = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 - s_1 c_2 c_3 & s_1 s_2 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 + c_1 c_2 c_3 & -c_1 s_2 \\ s_2 s_3 & s_2 c_3 & c_2 \end{bmatrix} \]

Solving \((\beta_1, \beta_2, \beta_3)\) from \(R\):

\[ \beta_2 = \cos^{-1} R_{33} \in [0, \pi] \text{ or } -\cos^{-1} R_{33} \]

\[ \beta_1 = \text{atan2}(R_{13}, -R_{23}) \text{ if } s_2 \geq 0 \text{ or } \text{atan2}(R_{13}, -R_{23}) + \pi \text{ otherwise} \]

\[ \beta_3 = \text{atan2}(R_{31}, R_{32}) \text{ if } s_2 \geq 0 \text{ or } \text{atan2}(R_{13}, -R_{23}) + \pi \text{ otherwise}. \]

There are two solutions in general and they merge when \(\beta_2 = 0\) or \(\pi\). We shall see that this corresponds to the singularity in the representation.
Euler Angle (Cont.)

There could be redefinition of the axis as well, e.g., the tool frame representation for the original Unimate controller (for PUMA robots):

\[
R_{03} = e^{\varepsilon \beta_1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} e^{\varepsilon \beta_2} e^{\varepsilon \beta_3}.
\]

In general: if \( p \) is a representation of \( R \), then \( R(p) \) is sometimes called the forward kinematics, and \( p(R) \) is called the inverse kinematics.