\( R \in SO(3) \)
\[
\gamma_i = \gamma_c + R \gamma_i c = \gamma_c + (r_c - r_c_i)
\]
\[
= \gamma_c + R(r_c - r_c_i)
\]
\[
= \gamma_c + R(r_c - r_c_i) + R P_i
\]
\[
R_c = 0 \quad R = I
\]
\[
= \gamma_c + R(R^T(r_c - r_c_i) + P_i)
\]
\[
= \gamma_c + R R^T (r_c - r_c_i) + P_i
\]
\[
\gamma_i(t_j) = \gamma_c(t_j) + R(t_j) P_i
\]
\[
\begin{pmatrix}
\gamma_c(t_k) & r_c(t_k) & 1 \\
\vdots & \vdots & \vdots \\
\gamma_n(t_k) & r_n(t_k) & 1
\end{pmatrix}
= \begin{pmatrix}
\gamma_c(t_1) & R(t_1) & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_c(t_m) & R(t_m) & 1 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
P_i & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & P_m
\end{pmatrix}
\]
Multiple robots with rigid grasp

Each arm is fully actuated.
- More actuated joints than DOF of mechanism.

\[
\begin{bmatrix}
\dot{\omega}_i \\
\dot{\mathbf{v}}_i \\
\mathbf{v}_i
\end{bmatrix} = \begin{bmatrix}
\mathbf{0} \\
\mathbf{P}_i \times \mathbf{I} \\
\mathbf{A}_i
\end{bmatrix} \begin{bmatrix}
\dot{\omega}_T \\
\dot{\mathbf{v}}_T \\
\mathbf{v}_T
\end{bmatrix}
\]

= \mathbf{J}_i(\mathbf{q}_i) \ddot{\mathbf{q}}_i

\[
\begin{bmatrix}
\mathbf{v}_1 \\
\vdots \\
\mathbf{v}_m
\end{bmatrix} = \begin{bmatrix}
\mathbf{J}_1(\mathbf{q}_1) & \mathbf{0} & \cdots & \mathbf{J}_m(\mathbf{q}_m)
\end{bmatrix} \begin{bmatrix}
\dot{\mathbf{q}}_1 \\
\vdots \\
\dot{\mathbf{q}}_m
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}_1 \\
\vdots \\
\mathbf{A}_m
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_T
\end{bmatrix}
\]

\[
\begin{cases}
\mathbf{v}_T = \mathbf{A}^+ \mathbf{J} \mathbf{q} \\
\mathbf{0} = \mathbf{A}^+ \mathbf{J} \mathbf{g}
\end{cases}
\]

\[\mathbf{A}^+ = (\mathbf{A} \mathbf{A}^T)^{-1} \mathbf{A}^T\]
\[ A = \begin{bmatrix}
  A_1^{-1} & -A_2^{-1} & \cdots & 0 \\
  A_1^{-1} & -A_3^{-1} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  A_1^{-1} & -A_m^{-1} & \cdots & 0
\end{bmatrix} \]

Manipulability Ellipsoid:

\[ \mathcal{E} = \{ \mathbf{v}_T : \mathbf{v}_T = \mathbf{J} \mathbf{\hat{q}}, \quad \| \hat{q} \| = 1 \} \]

\[ \mathcal{E} = \{ \mathbf{v}_T : \mathbf{v}_T = A^T \mathbf{J} \mathbf{\hat{q}}, \quad \| \mathbf{J} \mathbf{\hat{q}} \| = 0 \} \]

\[ \| \hat{q} \| = 1 \]
\[ V_T = A^+ J \hat{z} \]

\[ \hat{z} = \frac{1}{\sqrt{V(N(\hat{A}J))}} \]

\[ \hat{A}J \hat{z} = 0 \text{ basis of } \text{span} \{ \hat{A}J \hat{z} \} \]

\[ V_T = A^+ J \hat{A} J \hat{z} \]

\[ \hat{A}J \hat{z} \]

Ellipsoid is given by

SVD of \[ A^+ J \hat{A} J \lambda^{-\frac{1}{2}} \]
Multi-Finger Grasp

\[
\begin{align*}
\tau_i &= A_i \tau_i^T \\
\tau_i^T &= J_i(\theta_i) \dot{\theta}_i
\end{align*}
\]

\[
A_T = J \dot{q} + H \dot{w}
\]

Contact velocity

Columns of \(H_i\) are the directions of feasible motion at contact

Passive
Types of contacts

- Rigid grasp: \( H_i = 0 \)

- Sliding:

  \[ H_i \mathbf{W}_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{tang}} \\ y_{\text{tang}} \end{bmatrix} \]
  \[ 6 \times 2 \]

- Hinge:

  \[ H_i \mathbf{W}_c = \begin{bmatrix} h \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \]
  \[ 6 \times 1 \]

- Point contact without friction

  \[ H_i \mathbf{W}_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{tang}} \\ y_{\text{tang}} \end{bmatrix} \]
  \[ 6 \times 5 \]

- Point contact with friction

  \[ H_i \mathbf{W}_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \]
  \[ 6 \times 3 \]
Soft finger contact (with friction)

\[ \mathbf{H} = \begin{bmatrix} \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_x & \mathbf{h}_y \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \]

\[ \mathbf{W}_c = \mathbf{h}_x \mathbf{h}_y \]

Friction cone

Friction force \( = \mu f_n \)

\[ \text{Static coefficient of friction} \quad \tan \phi \quad \tan \alpha \]

\[ \mu f_n > f_t \quad \text{or} \quad \frac{f_t}{f_n} < \mu \]

If there is no motion, then

\[ \tan \phi < \tan \alpha \]

\[ \Leftrightarrow \phi < \alpha \in \text{Friction cone (FC) condition} \]
Back to anger kinematics:

\[ AV_T = J_D^g + HW \]

\[ \tilde{H} \tilde{A} V_T = (\tilde{H} J^g D) + \tilde{H} W \]

\[ \tilde{H} = \text{annihilator of } H \]

\[ GT \rightarrow \tilde{H} \]

\[ GT V_T = J_h \frac{\partial}{\partial R} \rightarrow \text{hand Jacobian} \]

\[ G_T \text{ is called the grasp map} \]

**Def:** \( G \) is called a task grasp if \( \text{dim}(C(G)) = 3 \) (equivalently, \( R(G) = 1R^6 \), or \( G \) is full rank).

**Def:** \((J_h G)\) is a manipulable grasp if \( R(G) \subset R(J_h) \).
Relationship to General Parallel Mechanism

\[ 0 = \tilde{A} V T = \tilde{A} T \hat{q} + \tilde{A} H \tilde{w} \]

Let \( x \in N(\tilde{A} H) \) suppose \( T_{cp} \) is singular (i.e., unstable singularity)

\[ \Delta = 0 \quad \tilde{A} H x = 0 \quad (N(\tilde{A}) = R(\tilde{A})) \]

\[ \Delta = 0 \quad \exists y \neq 0 \quad H x = A y \]

\[ \Delta = 0 \quad \tilde{H} A y = 0 \]

\[ \Delta = 0 \quad y \in N(\tilde{H} A) \]

\[ \therefore N(\tilde{A} H) \neq \{0\} \quad \Delta = 0 \quad N(G^T) \neq \{0\} \]

\underline{unstable singularity} \qquad \underline{unstable grasp}
Dual Perspective: Force/Faigue Balance

\[ f_T = A_1^T \phi_1 + \ldots + A_m^T f_m = A^T f_T \]

\[
\begin{bmatrix}
I - P_{ff}^T & 0 & 0
\end{bmatrix}
\]

\[ H^T f_T = 0 \]

\[ H^T f = 0 \Rightarrow f = \tilde{H}^T \eta \]

\[ f_T = \tilde{A}^T \tilde{H} \eta \]

Grasp map again

\[ N(G^T) \]

\[ R(G) \]
Revisit frictional contacts

Point contact without friction:

\[ H_i = \begin{bmatrix} \hat{n}_i \cdot \hat{r}_i \end{bmatrix} \]

\[ \gamma_i = f_{n_i} \]  

Scalar magnitude of normal force

\[ f_c = H_i \gamma_i \]  

\[ \gamma_i > 0 \]

Point contact with friction:

\[ H_i = \begin{bmatrix} 0 \mid 2 \end{bmatrix} \]

\[ \gamma_i = \frac{F_i}{3 \times 1} \]  

\[ f_c = H_i \gamma_i \]  

\[ \gamma_i \in (FC)_i \]

Soft finger contact:

\[ H_i = \begin{bmatrix} \hat{n}_i \cdot \hat{r}_i \mid 0 \mid 0 \end{bmatrix} \]

\[ \gamma_i = \begin{bmatrix} \dot{\gamma}_{n_i} \mid 0 \mid 0 \end{bmatrix} \]  

\[ \gamma_i \in (FC)_i \]
\[(FC)_G : \{ F \in \mathbb{R}^3 : \vec{F} - (\vec{F} \cdot \vec{n}) \vec{n} \leq \mu \tan \alpha \} \]

\[
\vec{F} \cdot \vec{n} > 0, \quad \vec{n} \text{ is the surface normal, into the object } \}

**Stable grasp**

**Example:**

Point contacts without friction

\[ f_1 = G \eta \]

\[ G = \begin{bmatrix} -a & b & -a & b \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \]

\[ f = \begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}, \quad \eta = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \]

If \( a, b \) are not both zero, \( G \) is full row rank

\[ \therefore N(G^T) = \{0\} \]