Robotics & Automation

Lecture 24

Order $N$ Inverse and Forward Dynamics

John T. Wen

April 23, 2007
Last Time

- Newton-Euler approach to dynamic model of open serial chain
Velocity/Acceleration/Force propagation:

\[ V_{i+1} = \phi_{i+1,i} V_i + H_{i+1} \dot{\theta}_{i+1} \]
\[ \alpha_{i+1} = \phi_{i+1,i} \alpha_i + H_{i+1} \ddot{\theta}_{i+1} + a_{i+1} \]
\[ f_i = \phi_{i+1,i} f_{i+1} + M_i \alpha_i + b_i \]
\[ \tau_i = H_i^* f_i \]
\[ V_0 = 0, \quad \alpha_0 = \begin{bmatrix} 0 \\ \vec{g} \end{bmatrix} \]
\[ f_{N+1} = f_{ext} \]

Relationship to Lagrange-Euler formulation

\[ \tau = H^T f = H^T \Phi^T B f_{ext} + H^T \Phi^T (M \alpha + b) \]
\[ = (H^T \Phi^T M \Phi H) \ddot{\theta} + H^T \Phi^T (M \Phi a + b) + H^T \Phi^T M \Phi E \alpha_0 + H^T \Phi^T B f_{ext}. \]

Order N Inverse Dynamics: Recursive Newton-Euler formulation (2-sweep algorithm). See MATLAB code on-line run_invdyn.m.
Today: Order $N$ Forward Dynamics

Problem: Given $(\theta, \dot{\theta}), \tau,$ and $f_{ext},$ find $\ddot{\theta}.$

From Lagrange-Euler, we can just use

$$\ddot{\theta} = M(\theta)^{-1}( - C(\theta, \dot{\theta}) \dot{\theta} - G(\theta) + \tau - J^T(\theta) f_{ext} ).$$

However $M^{-1}$ is an $N^3$ type of operation, therefore inefficient for large number of degrees of freedom. We shall see that it can also be computed in $O(N)$. 
Order $N$ Forward Dynamics

**Step 1:** Forward sweep of $V$ equation. Save $a_i$ and $b_i$.

**Step 2:** Backward sweep.
First consider the force Propagation: Write $f_i$ in terms of $\alpha_i$:

\[
f_i = \phi^T_{i+1,i} f_{i+1} + M_i \alpha_i + b_i
\]

\[
= \phi^T_{i+1,i} (\phi^T_{i+2,i+1} f_{i+2} + M_{i+1} \alpha_{i+1} + b_{i+1}) + M_i \alpha_i + b_i.
\]

Now substitute the acceleration propagation for $\alpha_{i+1}$:

\[
f_i = \phi^T_{i+2,i} f_{i+2} + (\phi^T_{i+1,i} M_{i+1} \phi_{i+1,i} + M_i) \alpha_i + \phi^T_{i+1,i} H_{i+1} \ddot{\theta}_{i+1}
\]
\[
+ \phi^T_{i+1,i} (M_{i+1} a_{i+1} + b_{i+1}) + b_i.
\]

In general, we hypothesize the following relationship between $\alpha_i$ and $f_i$:

\[
f_i = P_i \alpha_i + \beta_i
\]

where $P_i$ is called the articulated body inertia, and $\beta_i$ only depends on the velocities.
We first solve for $\ddot{\theta}_{i+1}$.

\[
\tau_{i+1} = H^T_{i+1} f_{i+1}
\]
\[
= H^T_{i+1} (P_{i+1} \alpha_{i+1} + \beta_{i+1})
\]
\[
= H^T_{i+1} (P_{i+1} (\phi_{i+1,i} \alpha_i + H_{i+1} \ddot{\theta}_{i+1} + a_{i+1}) + \beta_{i+1})
\]
\[
= H^T_{i+1} P_{i+1} H_{i+1} \ddot{\theta}_{i+1} + H^T_{i+1} P_{i+1} \phi_{i+1,i} \alpha_i + H^T_{i+1} (P_{i+1} a_{i+1} + \beta_{i+1}).
\]

Now we can solve for $\ddot{\theta}_{i+1}$:

\[
\ddot{\theta}_{i+1} = \frac{(\tau_{i+1} - H^T_{i+1} P_{i+1} \phi_{i+1,i} \alpha_i - H^T_{i+1} (P_{i+1} a_{i+1} + \beta_{i+1}))}{H^T_{i+1} P_{i+1} H_{i+1}}.
\]
We can now obtain the iteration on $P_i$ and $\beta_i$:

\[
\begin{align*}
  f_i &= P_i \alpha_i + \beta_i \\
  &= \phi_{i+1,i}^T \left( P_{i+1} \alpha_{i+1} + \beta_{i+1} \right) + M_i \alpha_i + b_i \\
  &= \phi_{i+1,i}^T \left( P_{i+1} \left( \phi_{i+1,i} \alpha_i + H_{i+1} \ddot{\theta}_{i+1} + a_{i+1} \right) + \beta_{i+1} \right) + M_i \alpha_i + b_i \\
  &= \left( M_i + \phi_{i+1,i}^T P_{i+1} \phi_{i+1,i} - \frac{\phi_{i+1,i}^T P_{i+1} H_{i+1} H_{i+1}^T P_{i+1} \phi_{i+1,i}}{H_{i+1}^T P_{i+1} H_{i+1}} \right) \alpha_i \\
  \quad \text{(} P_i = f_P(P_{i+1}) \right) \\
  + \left( \phi_{i+1,i}^T P_{i+1} \alpha_{i+1} + \beta_{i+1} \right) + b_i + \\
  \phi_{i+1,i}^T P_{i+1} H_{i+1} \left( \tau_{i+1} - H_{i+1}^T \left( P_{i+1} \alpha_{i+1} + \beta_{i+1} \right) \right) \\
  \quad \text{(} \beta_i = f_\beta(P_{i+1}, \beta_{i+1}) \right).
\end{align*}
\]

Boundary condition:

\[
\begin{align*}
  f_N &= \phi_{N+1,N}^T f_{\text{ext}} + M_N \alpha_N + b_N \\
  &= \left( M_N \alpha_N + (\phi_{N+1,N}^T f_{\text{ext}} + b_N) \right) \quad \text{(} \beta_N \right).
\end{align*}
\]
So the second sweep is a backward sweep from tip to base to find $P_i$ and $\beta_i$. 

**Step 3** Final forward sweep.

\[
\ddot{\theta}_{i+1} = \frac{\tau_{i+1} - H_{i+1}^T P_{i+1} \phi_{i+1, i} \alpha_i - H_{i+1}^T (P_{i+1} a_i + \beta_{i+1})}{H_{i+1}^T P_{i+1} H_{i+1}}
\]

\[
\alpha_{i+1} = \phi_{i+1, i} \alpha_i + H_{i+1} \ddot{\theta}_{i+1} + a_{i+1}, \quad \alpha_0 = \begin{bmatrix} 0 \\ \vec{g} \end{bmatrix}.
\]

Since there is a fixed number of operations for each link, the total number of operations is of order $N$. 
Consider the inverse dynamics problem: $\tau = M\ddot{\theta}$. In general, for

$$
\begin{align*}
z_{\begin{array}{c} N \\ 1 \end{array}} &= \begin{bmatrix} A \\ N \times N \end{bmatrix} x_{\begin{array}{c} N \\ 1 \end{array}}
\end{align*}
$$

or

$$
z_i = \sum_{j=1}^{N} a_{ij}x_j,
$$

there are $N$ multiplications and $N - 1$ additions, for a total of $N^2$ multiplications and $N(N - 1)$ additions.

But we know more about the structure of $M$:

$$
M = H^T \Phi^T \mathcal{M} \Phi H,
$$

where $H^T \Phi^T$ is upper triangular (anticausal), $\mathcal{M}$ is block diagonal, and $\Phi H$ is lower triangular (causal). Furthermore, entries of $\Phi$ satisfy the group property:

$$
\phi_{ij} \phi_{jk} = \phi_{ik}, \quad \phi_{ii} = I.
$$
Now consider $z = Ax$ again where $A$ is causal and has the group structure. Then

\[
\begin{align*}
    z_1 &= \phi_{11} x_1 \\
    z_2 &= \phi_{22} x_2 + \phi_{21} x_1 = \phi_{22} x_2 + \phi_{21} \phi_{11} x_1 \\
    z_3 &= \phi_{33} x_3 + \phi_{32} x_2 + \phi_{31} x_1 \\
         &= \phi_{33} x_3 + \phi_{32} (\phi_{22} x_2 + \phi_{21} x_1) \\
         &= \phi_{33} x_3 + \phi_{32} z_2 \\
    \vdots \\
    z_N &= \phi_{NN} x_N + \phi_{N,N-1} z_{N-1}.
\end{align*}
\]

This only requires $2N - 1$ multiplications and $N - 1$ additions. Similar procedure also holds for anticausal multiplications. Therefore, the inverse dynamics can be performed in order $N$ operations.

What about forward dynamics?
Kalman Filter as Whitening Filter

Input sequence is white: $E[w_k] = 0$ and $E[w_k w_j^T] = \delta_{kj} M_k$.

System is linear time varying:

$$
x_k = \phi_{k+1,k}^T x_{k+1} + w_k \\
y_k = H_k^T x_k.
$$

Write $y = [y_1, ..., y_N]^T$, $w = [w_1^T, ..., w_N^T]^T$. Then

$$
E[y y^T] = H^T \phi^T M \phi H = M(\theta),
$$

which is the mass matrix.
Now relate $y$ to $\xi$ (another white sequence, called the innovation sequence): $\xi = \mathcal{K}y$ or $y = \mathcal{K}^{-1}\xi$. Then

$$M(\theta) = E[yy^T] = \mathcal{K}^{-1}E[\xi\xi^T]\mathcal{K}^{-T}.$$  

The inverse is then

$$M^{-1}(\theta) = \mathcal{K}^T E[\xi\xi^T]^{-1} \mathcal{K}.$$

Since $\mathcal{K}$ is causal and has a group structure, $M^{-1}(\theta)\tau$ may be computed in order $N$ steps.
What is Kalman Filter?

Kalman filter estimates the state $x_k$ based on $y_k$:

\[
\hat{x}_k = \phi_{k+1,k}^T \hat{x}_{k+1} + L_{k+1} \left( y_{k+1} - H_{k+1}^T \hat{x}_{k+1} \right)
\]

Kalman Gain

\[
\xi_k = y_k - H_k^T \hat{x}_k.
\]

Let $P_k = E\left[ (x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \right]$ be the error covariance. Then the Kalman gain is

\[
L_{k+1} = \frac{\phi_{k+1,k}^T P_{k+1} H_{k+1}}{H_{k+1}^T P_{k+1} H_{k+1}}
\]

and $P_k$ satisfies the recursion (discrete Riccati Equation):

\[
P_k = M_k + \phi_{k+1,k}^T P_{k+1} \phi_{k+1,k} - L_{k+1} H_{k+1}^T P_{k+1} H_{k+1} L_{k+1}^T.
\]

The innovation $\xi_k = y_k - \hat{y}_k = H_k^T (x_k - \hat{x}_k)$ is white with covariance

\[
E[\xi_k \xi_j^T] = \delta_{kj} \left( H_k^T P_k H_k \right)_{D_k}.
\]
Rewrite the Kalman filter with $y$ as input and $\xi$ as output:

$$
\hat{x}_k = \left( \phi_{k+1,k}^T - L_{k+1} H_{k+1}^T \right) \hat{x}_{k+1} + L_{k+1} y_{k+1}
$$

$$
\xi_k = -H_k^T \hat{x}_k + y_k.
$$

Note that $\psi_{k+1,k}$ also satisfies the group property. Then

$$
\xi = (I - H^T \psi^T L) y.
$$

Finally, $M^{-1}(\theta)$ can be factor as

$$
M^{-1}(\theta) = \left( I - H^T \psi^T L \right)^T D^{-1} \left( I - H^T \psi^T L \right),
$$

which means $M^{-1} \tau$ can be performed in order $N$ operations.