$$\dot x = \cos\theta \, v \qquad\qquad \dot x_1 = u_1$$

$$\dot y = \sin\theta \, v \quad \longrightarrow \quad \dot x_2 = u_2$$

$$\dot\theta = w \qquad\qquad \dot x_3 = x_1 u_2$$

$$z_1 = \cos\theta \, x + \sin\theta \, y$$

$$\dot z_1 = v + \underbrace{(-\sin\theta \, x + \cos\theta \, y)}\overset{w}{\boxed{\dot\theta}}$$

$$z_2 = -\sin\theta \, x + \cos\theta \, y$$

$$\dot z_2 = -(\cos\theta \, x + \sin\theta \, y) w$$

$$\dot z_2 = -z_1 w$$

$$\dot z_1 = v + z_2 w$$

$$z_3 = \theta$$

$$\dot z_3 = w$$

$$\begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix} \longleftrightarrow \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

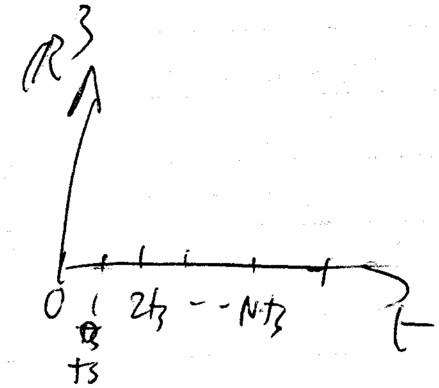
$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\underline{\underline{\dot{x}_3}} = x_1 u_2$$

$$\boxed{\dot{x} = g(x)\,u}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$x_k \triangleq x(\underline{\underline{k\, t_s}})$$

$$\frac{x_{k+1} - x_k}{t_s} \cong g(x_k)\,u_k \qquad \begin{bmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \end{bmatrix}$$

$$\boxed{x_{k+1} = \underbrace{x_k + t_s\, g(x_k)\, u_k}_{f(x_k,\, u_k)} \; ; \; x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad x_N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$J = \sum_{k=0}^{N-1} \underbrace{\left(u_{1k}^2 + u_{2k}^2 \right)}_{\|u_k\|^2}$$

$$H_k = \|u_k\|^2 + \lambda_{k+1}^T \left(x_k + t_s g(x_k) u_k\right)$$

Solve for u_k:

$$\frac{\partial H_k}{\partial u_k} = 0 \qquad 2u_k + t_s g^T(x_k)\lambda_{k+1} = 0$$

$$u_k = -\frac{1}{2} t_s g^T(x_k)\lambda_{k+1}$$

$$\underbrace{\lambda_k}_{3\times 1} = \left(\frac{\partial H_k}{\partial x_k}\right)^T = \underbrace{\lambda_{k+1}}_{3\times 1} + \frac{t_s\,\partial g(x_k)}{\partial x_k}\lambda_{k+1}$$

$$\lambda_{1k} = \frac{\partial H_k}{\partial x_{1k}} = \lambda_{1k+1} + t_s \frac{\partial\left(\lambda_{k+1}^T g(x_k) u_k\right)}{\partial x_{1k}}$$

$\lambda_{3k+1} u_{2k}$

$$\lambda_{2k} = \frac{\partial H_k}{\partial x_{2k}} = \lambda_{2k+1}$$

$$\lambda_{3k} = \frac{\partial H_k}{\partial x_{3k}} = \lambda_{3k+1}$$

u_{1k}

u_{2k}

$x_{1k} u_{2k}$

$\lambda_{1k+1} u_{1k} + \lambda_{2k+1} u_{2k} + \lambda_{3k+1} x_{1k} u_{2k}$

$$\begin{cases} \lambda_{2k+1} = \lambda_{2k} \\ \lambda_{3k+1} = \lambda_{3k} \\ \lambda_{1k+1} = \lambda_{1k} - t_s \lambda_{3k} \left(-\frac{1}{2} t_s\right)\big(\underbrace{\lambda_{2k+1}}_{=\lambda_{2k}} + x_{1k} \underbrace{\lambda_{3k+1}}_{=\lambda_{3k}}\big) \\ \quad = \lambda_{1k} + \frac{t_s^2}{2} \lambda_{3k} (\lambda_{2k} + x_{1k} \lambda_{3k}) \end{cases}$$

x_0

x_N

$$x_{k+1} = x_k - \frac{t_s^2}{2} g(x_k)\, g^T(x_k)\, \lambda_{k+1}$$

phi.m

$$\begin{matrix} x_0 \\ \boxed{\lambda_0} \end{matrix} \rightarrow \begin{matrix} x_1 \\ \lambda_1 \end{matrix} \rightarrow \begin{matrix} x_2 \\ \lambda_2 \end{matrix} \dashrightarrow x_N$$

given

$$\phi(\lambda_0) - x_N = 0$$

function $xN = phi(lambda0)$

$x_0 \rightarrow$

fsolve

or

fminunc

$$\|\phi(\lambda_0) - x_N\|^2$$

λ_0

$$x_{k+1} = x_k - \frac{t_s^2}{2} g(x_k)\, g^T(x_k)\, \lambda_{k+1}$$

x_0 given
x_N given

$$\lambda_k = \lambda_{k+1} + t_s \begin{bmatrix} \cdot & 0 & u_{2k} \\ \cdot & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_{k+1}$$

$$\lambda_{k+1} = \begin{bmatrix} \lambda_{1k+1} \\ \lambda_{2k+1} \\ \lambda_{3k+1} \end{bmatrix}$$

Define a function phi.m

fsolve

fmin

$$\phi_N(\lambda_0) \rightarrow \| \hat{x}_N - x_N \|^2$$

$$\lambda_0 = \begin{pmatrix} \lambda_{01} \\ \lambda_{02} \\ \lambda_{03} \end{pmatrix}$$

$\hat{x}_N$

$$\phi_N(\lambda_0) - x_N = 0$$

$$u_k = -\frac{1}{2} t_s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_{1k} \end{bmatrix} \begin{bmatrix} \lambda_{1k+1} \\ \lambda_{2k+1} \\ \lambda_{3k+1} \end{bmatrix}$$

$$= -\frac{1}{2} t_s \begin{bmatrix} \lambda_{1k+1} \\ \lambda_{2k+1} + x_{1k} \lambda_{3k+1} \end{bmatrix}$$

$$\lambda_k = \begin{bmatrix} 1 & 0 & t_s u_{2k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_{k+1}$$

$$-\frac{1}{2} t_s^2 (\lambda_{2k+1} + x_{1k} \lambda_{3k+1})$$

$$\begin{pmatrix} x_0 \\ \lambda_0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} - - - \rightarrow \begin{pmatrix} x_N \\ \lambda_N \end{pmatrix}$$

$$\lambda_{k+1} = \begin{pmatrix} 1 & 0 & -t_s u_{2k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda_k$$

$$x_{k+1} = f(x_k, u_k) \quad ; \; x_0$$

Suppose you are given

$$\begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \to \quad \begin{aligned} x_1 &= f(x_0, u_0) \\ x_2 &= f(x_1, u_1) \\ &\vdots \\ x_N &= f(x_{N-1}, u_{N-1}) \\ &= \psi(x_0 ; (u_0, u_1, \dots, u_{N-1})) \end{aligned}$$

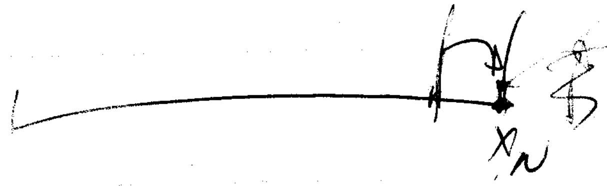
$$\min_{\underbrace{(u_0, \dots, u_{N-1})}_{40}} \|\psi(x_0, (u_0, \dots, u_{N-1})) - \overset{\text{given}}{x_N}\|^2 + \underbrace{\|u_0\|^2 + \|u_1\|^2 + \dots + \|u_{N-1}\|^2}$$

2×1

fmincon

$$\min \; \|u_0\|^2 + \dots + \|u_{N-1}\|^2 \qquad \underline{u}^T \underline{u}$$

$$\text{subject to} \quad \psi(x_0 ; \underbrace{(u_0, \dots, u_{N-1})}_{\underline{u}}) - \overset{\text{given}}{x_N} = 0$$



$$\underline{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

40×1