Optimal Control

Lecture 20
Control Lyapunov Function, Optimal Estimation

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Ref: Papers by R. Freeman (on-line), B&H Ch. 12
Outline

- Summary of Control Lyapunov Function and example
- Introduction to optimal estimation
Feedback control of nonlinear systems

- Stabilization: linearization, feedback linearization, etc.
- Optimal control: HJB equation

**Optimality implies stability:**
Value function \( V^* (x) \) (solution of HJB equation) for a "meaningful" optimal control problem

\[
J = \int_{0}^{\infty} q(x) + r(x,u) \, dt.
\]

\( q \) is positive definite in \( x \) and \( r \) is positive definite in \( u \) for all \( x \), is a Lyapunov function.

\[
\dot{V}^* = \frac{\partial V^*}{\partial x} (f(x) + g(x)u) \leq -(q(x) + r(x,u^*(x)))
\]

which is negative definite.

Converse question: **Does stability imply optimality?**
A simple example

Consider

\[ \dot{x} = -x^3 + u \]

Feedback linearization: \( u = x^3 - 2x \), cancels beneficial nonlinearity and gives positive feedback for large \( x \) (very non-robust!).

Optimal control (for \( L(x, u) = x^2 + u^2 \)): \( u^* = k(x) = -x^3 - x\sqrt{x^4 - 2x^2} + 2 \). Control effort drops off for large \( x \) so take advantage of the beneficial \(-x^3\) term. Optimal control is much more attractive. However ... HJB tough to solve in general!
Control Lyapunov Function Approach

Control Lyapunov function (clf) approach provides a computable middle ground: find stabilizing control laws with “meaningful” optimality property.

Control Lyapunov function: \( V(x) \) p.d. such that
\[
\min_u \left( \frac{\partial V}{\partial x} (f(x) + g(x)u) \right) < 0.
\]
Lyapunov function for any stabilizing control law is also a clf.

Pointwise mini-norm control law (for any p.d. \( \sigma(x) \)):
\[
k(x) = \arg \min \left\{ \|u\| : \frac{\partial V}{\partial x} (f(x) + g(x)u) \leq -\sigma(x) \right\}.
\]

Then there exist (see Freeman and Kokotovic, SIAM J. on Opt. & Control, 1996) \( q(x) \) p.d., \( r(x,u) \) p.d. for all \( x \), such that \( k(x) \) is the optimal solution.

clf can be used to construct stabilizing control laws with meaningful optimality property.
clf approach may be applied locally (near equilibrium) also.
Choose $\sigma(x) = -2x^2$, then pointwise min-norm control law is

$$u = k(x) = \begin{cases} 
  x^3 - 2x & x^2 < 2 \\
  0 & x^2 \geq 2
\end{cases}.$$
Sontag universal control law

Given positive definite $q(x)$, choose $\sigma(x)$ to be

$$\sigma(x) = \sqrt{(L_f V)^2 + q(L_g V)^2}$$

$L_f V$ is the Lie derivative of $V$ along $f$: $L_f V := \frac{dV}{dt} f(x)$. Sontag universal control law for the single input case (continuous everywhere except possibly at $x = 0$):

$$u = \begin{cases} 
-(L_f V + \sigma(x))/(L_g V) & L_g V \neq 0 \\
0 & L_g V = 0
\end{cases}.$$ 

The corresponding optimal control may have a different $q$. However, if $V$ and $V^*$ have the same level surfaces then $u$ is optimal. Motivation: the optimal control $u^*$ corresponding to $L(x, u) = q(x) + u^2 / 2$ satisfies

$$u = -(L_f V^* + \sqrt{(L_f V^*)^2 + q(L_g V^*)^2})/(L_g V^*).$$
Simple example cont.

For the example: $u = x^3 - x\sqrt{x^4} + 1$. 
**Least Square Estimator**

Given the measurement vector $z$ and a hypothesized relationship, $z = Hx + n$, we would like to estimate $x$.

We pose this as an optimization problem: minimize

$$J_z = \frac{1}{2} e_z^T e_z$$

where $e_z$ is the output estimation error (output residue)

$$e_z = z - \hat{z} = z - H\hat{x} = Hx + n - H\hat{x}.$$

Solution: $\partial J / \partial \hat{x} = 0$.
In general: $\hat{x} = H^+ z$.
If $H$ is full column rank: $H^+ = (H^TH)^{-1}H^T$.
Weighted least square: $J = \frac{1}{2} e_z^T W e_z$.

$$\hat{x} = (H^T W H)^{-1} H^T W z.$$
Least Square Estimator

Estimation error:
\[
J_e = \frac{1}{2} n^T (I - H(H^T H)^{-1} H^T) n.
\]
\[
J_e = \frac{1}{2} n^T H(H^T H)^{-1} (H^T H)^{-1} H^T n.
\]

Examples:
- estimating a scalar constant
- fitting data to a cubic polynomial
- state estimation in linear time invariant systems
- parameter estimation