Multivariable Control

Lecture 06
Multivariable Poles and Zeros

John T. Wen

September 20, 2004

Ref: Sec. 3.6 of text.
SISO transfer function: \( G(s) = \frac{n(s)}{d(s)} \) (no common factors between \( n(s) \) and \( d(s) \)).

- **Poles**: values of \( s \) such that \( d(s) = 0 \).
- **Zeros**: values of \( s \) such that \( n(s) = 0 \).

Watch out for pole/zero cancellation in cascaded systems!

\[
\begin{array}{ccc}
\text{u} & \xrightarrow{\frac{s - 1}{s + 1}} & \xi \\
& & \xrightarrow{\frac{1}{s - 1}} & y
\end{array}
\]

(Find the state space realization.)
For MIMO: $G(s) = D + C(sI - A)^{-1}B$ (rational matrix: matrix of rational functions). We will write $G(s)$ as

$$G(s) = D^{-1}(s)N_L(s) = N_R(s)D^{-1}(s)$$

where $D_L, N_L, N_R, D_R$ are polynomial matrices.
Some Definitions

Definition
\[ \mathbb{R}^{p \times m}[s] = p \times m \text{ polynomial matrix of } s \text{ with coefficients in } \mathbb{R} \]
\[ \mathbb{R}^{p \times m}(s) = p \times m \text{ rational matrix of } s \text{ with coefficients in } \mathbb{R} \]
\[ \mathbb{R}_p(s) = \text{ proper rational functions} \]
\[ \mathbb{R}_{p,o}(s) = \text{ strictly proper rational functions} \]

Definition
The normal rank of \( G \in \mathbb{R}^{p \times m}[s] \) is the maximum rank of \( G \) over all \( s \in \mathbb{C} \).

Definition (Generalization of the concept of scalar constants in \( \mathbb{R}[s] \))
\( G \in \mathbb{R}^{k \times k}[s] \) is unimodular if \( G^{-1} \in \mathbb{R}^{k \times k}[s] \).

Fact: \( G \) is unimodular iff \( G^{-1} \in \mathbb{R}[s] \) iff \( \det G = \text{constant} \).

\[
\begin{bmatrix}
1 & s + 1 \\
0 & 2 \\
\end{bmatrix}
\]
is unimodular, but
\[
\begin{bmatrix}
1 & 2 \\
0 & s + 1 \\
\end{bmatrix}
\]
is not.
Elementary Operations

- row/column
- interchange
- scaling
- scaling and addition

row: pre-multiplication
column: post-multiplication

All elementary operations are unimodular.
Let $M \in \mathbb{R}^{p \times m}[s]$ with rank($M$) $\leq \min(p,m)$. Then $M(s)$ can be transformed using elementary operations to

$$U(s)M(s)V(s) = S(s) = \begin{bmatrix} \text{diag}\{\varepsilon_1(s), \ldots, \varepsilon_r(s)\} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\varepsilon_i(s)$ is monic and $\varepsilon_i|\varepsilon_{i+1}$ ($\varepsilon_i$ divides $\varepsilon_{i+1}$). $\{\varepsilon_i\}$ are called the invariant factors of $M(s)$. 

**Smith Form**
Smith-McMillan Form

Let $G \in \mathbb{R}^{p \times m}(s)$. $G$ can be written as $G(s) = \frac{N(s)}{d(s)}$ where $N \in \mathbb{R}^{p \times m}[s]$ and $d \in \mathbb{R}[s]$ (l.c.m. of all denominators of $G_{ij}(s)$). Reduce $N(s)$ to its Smith Form, then

$$U(s)G(s)V(s) = U(s)N(s)V(s)/d(s) = S(s)/d(s).$$

Reduce $\sigma_i(s)/d(s)$ to lowest terms:

$$\frac{\sigma_i(s)}{d(s)} = \frac{\varepsilon_i(s)}{\psi_i(s)}.$$

Then

$$U(s)G(s)V(s) = \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\varepsilon_r(s)}{\psi_r(s)} \end{bmatrix}.$$
zero polynomial: \( z(s) = \varepsilon_1(s) \cdots \varepsilon_r(s) \)\). Roots are called transmission zeros of \( G(s) \).

pole polynomial: \( p(s) = \psi_1(s) \cdots \psi_r(s) \)\). Roots are called poles of \( G(s) \). \( \partial p \) (order of \( p(s) \)) is called the McMillan degree of \( G(s) \) (i.e., number of poles).

If \( G \) is square:
- transmission zeros are also the roots of \( \det G(s) = 0 \)
- if, in addition, \( G(\infty) \) is invertible, poles of \( G^{-1}(s) \) are the transmission zeros of \( G(s) \)
Example

Consider

\[ G(s) = \begin{bmatrix} \frac{s+1}{s+3} & \frac{s-1}{s+4} \\ 0 & \frac{1}{(s+1)(s+3)} \end{bmatrix} \]

Smith-McMillan Form:

\[ G_{SM}(s) = \begin{bmatrix} \frac{1}{(s+1)(s+3)(s+4)} & 0 \\ 0 & \frac{(s+1)(s+4)}{(s+3)} \end{bmatrix} \]

Clearly, there is no pole/zero cancellation for the \(-1\) and \(-4\) modes since these poles and zeros occur in different channels.
Write the Smith-McMillan Form of $G(s)$ as

$$U(s)G(s)V(s) = E(s)\Psi_r^{-1}(s) = \Psi_{\ell}^{-1}(s)E(s).$$

Then $G(s)$ can be written as

$$G(s) = \frac{(U(s)^{-1}E(s))(V(s)\Psi_{\ell}(s))^{-1}}{N_r(s)} = \frac{(\Psi_{\ell}(s)U(s))^{-1}(E(s)V^{-1}(s))}{D_r(s)}.$$
Coprime MFD

MIMO transfer matrix: \( G(s) = N_r(s)D_r^{-1}(s) = D_r^{-1}(s)N_l(s) \) where \( N_r, D_r, N_l, D_l \) are polynomial matrices (matrix fraction description, MFD). \( D_r \) is \( m \times m \), \( D_l \) is \( p \times p \), \( N_r \) is \( p \times m \), \( N_l \) is \( m \times p \).

Coprime MFD:

\((N_r, D_r)\) are right coprime: If there are polynomial matrices \( N_{r1}, D_{r1}, R \) such that

\[ N_r = N_{r1}R, \quad D_r = D_{r1}R \]

(so that \( N_rD_r^{-1} = N_{r1}D_{r1}^{-1} \)), then \( R \) must be unimodular, i.e., \( R \) and \( R^{-1} \) are both polynomial matrices (like nonzero constants for SISO systems).

\((D_l, N_l)\) are left coprime: If there are polynomial matrices \( D_{l1}, N_{l1}, L \) such that

\[ D_l = LD_{l1}, \quad N_l = LN_{l1} \]

(so that \( D_l^{-1}N_l = D_{l1}^{-1}N_{l1} \)), then \( L \) must be unimodular.
### Condition for Coprimeness

Row reduce
\[
\begin{bmatrix}
N_r(s) \\
D_r(s)
\end{bmatrix}
\begin{bmatrix}
R(s) \\
0
\end{bmatrix}
\]
(i.e., find an unimodular matrix \(X\) such that
\[
X(s)
\begin{bmatrix}
N_r(s) \\
D_r(s)
\end{bmatrix}
= \begin{bmatrix}
R(s) \\
0
\end{bmatrix}
\]. Then \((N_r,D_r)\) is right coprime if and only if \(R(s)\) is unimodular.

Column reduce
\[
\begin{bmatrix}
D_l(s) & N_l(s)
\end{bmatrix}
\begin{bmatrix}
L(s) \\
0
\end{bmatrix}
\]
(i.e., find an unimodular matrix \(Y\) such that
\[
\begin{bmatrix}
D_l(s) & N_l(s)
\end{bmatrix}
Y(s) = \begin{bmatrix}
L(s) \\
0
\end{bmatrix}
\]. Then \((D_l,N_l)\) is left coprime if and only if \(L(s)\) is unimodular.
Row and Column Reductions

To row reduce \[
\begin{bmatrix}
N_r \\
D_r
\end{bmatrix}
\], first find its Smith Form \[
U
\begin{bmatrix}
N_r \\
D_r
\end{bmatrix}
V = \Sigma,
\]
then
\[
U
\begin{bmatrix}
N_r \\
D_r
\end{bmatrix} = \Sigma V^{-1} =
\begin{bmatrix}
R \\
0
\end{bmatrix}.
\]

Similarly for column reduction.

MATLAB Tools

Use \texttt{smith} command in MAPLE (in MATLAB Symbolic toolbox) to transform
\[
\begin{bmatrix}
N_r(s) \\
D_r(s)
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
D_z(s) & N_z(s)
\end{bmatrix}
\]
to the Smith Form.
Bezout Identity

If \((N_r, D_r)\) is right coprime, then there exist polynomial matrices \((U_r, V_r)\) such that

\[ U_r N_r + V_r D_r = I. \]

If \((D_\ell, N_\ell)\) is left coprime, then there exist polynomial matrices \((V_\ell, U_\ell)\) such that

\[ D_\ell V_\ell + N_\ell U_\ell = I. \]

**Generalized Bezout Identity**

If \((N_r, D_r)\) are right coprime and \((D_\ell, N_\ell)\) are left coprime, then there exist polynomial matrices \((U_r, V_r, U_\ell, V_\ell)\) such that

\[
\begin{bmatrix}
U_r & V_r \\
D_\ell & -N_\ell
\end{bmatrix}
\begin{bmatrix}
N_r & V_\ell \\
D_r & -U_\ell
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

\((U_r, V_r, U_\ell, V_\ell)\) may be found using \texttt{smith} function in MATLAB symbolic toolbox.
Computation of Coprime MFD

Given a right MFD for \( G(s) = N_r(s)D_r^{-1}(s) \) (e.g., \( G(s) = \frac{N(s)}{d(s)} \)) where \( d(s) \) is the least common multiple of all denominators of \( G(s) \). Extract the greatest common right divisor, \( R \), for \( (N_r, D_r) \). Then \( N_r(s) = N_r(s)R^{-1}(s) \) and \( D_r(s) = D_r(s)R^{-1}(s) \) form a right coprime MFD for \( G(s) \).

Similarly, to find the left coprime MFD, start with any left MFD \( (D_\ell, N_\ell) \). Extract the greatest common left divisor, \( L \). Then \( D_\ell = L^{-1}D_\ell \) and \( N_\ell = L^{-1}N_\ell \) form a left coprime MFD.

Coprime MFD can be used in pole/zero characterization, state space realization and polynomial based control design.

Example: \( G_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{1}{s+1} \end{bmatrix} \) and \( G_2(s) = \begin{bmatrix} \frac{s+1}{s+3} & \frac{s-1}{s+4} \\ 0 & \frac{1}{(s+1)(s+3)} \end{bmatrix} \).
MIMO Pole/Zero: Characterization through MFD

Given coprime MFD’s for \(G(s): (N_r(s), D_r(s))\) and \((D_ℓ(s), N_ℓ(s))\).

Poles: values of \(s\) such that \(\det D_r(s) = \det D_ℓ(s) = 0\).

If \((A, B, C, D)\) is a minimal realization of \(G(s)\) (state dimension, \(n = \#\) of poles), then poles are also given by the roots of \(\det(sI - A) = 0\) (characteristic equation), i.e., eigenvalues of \(A\). (Note that \(G(s) = C\text{Adj}(sI - A)B/\det(sI - A) + D\).)

Transmission Zeros: values of \(s\) such that \(N_r(s)\) (or \(N_ℓ(s)\)) loses rank.

Blocking zeros: values of \(s\) such that \(G(s) = 0\).

Blocking zeros \(\subset\) Transmission zeros

Pole/zero cancellation in MIMO system may not be obvious, e.g., consider \(K(s)G(s)\) below

\[
G(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{1}{s+2} \\
\frac{2}{s+2} & \frac{1}{s+1}
\end{bmatrix}, \quad K(s) = \begin{bmatrix}
\frac{s+2}{s-\sqrt{2}} & -\frac{s+1}{s-\sqrt{2}} \\
0 & 1
\end{bmatrix}.
\]
Interpretation of poles and zeros

If $s_o$ is a pole of $G(s)$, then there exists an initial condition $x_o$ such that the zero input response of $G$ is $e^{s_o t} C x_o$ (choose $x_o$ to be the eigenvector of $A$ corresponding to $s_o$).

If $z_o$ is a transmission zero and $p \geq m$ (i.e., $G$ is square or tall), then there exist $u_o$ and $x_o$ such that the output under the input $u(t) = u_o e^{z_o t}$ and initial state $x_o$ is zero (certain input direction is blocked).

If $z_o$ is a transmission zero and $p \leq m$ (i.e., $G$ is square or fat), then there exist $v_o$ and $x_o$ such that under any input of the form $u_o e^{z_o t}$ and initial state $x_o$, output $y(t)$ is orthogonal to $v_o$, i.e., $v_o^T y(t) = 0$ (certain output direction cannot be achieved with any input).

If $z_o$ is a blocking zero, then $G(s) u_o e^{z_o t} = 0$ for all $u_o$.

MATLAB commands: pole, tzero
Example

\[ G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} \]

poles: \{-1,-1,-2,-2\} zero: -1.5

Evaluate \( G \) at the transmission zero:

\[ G(-1.5) = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \]

zero input direction: \( u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

unreachable output direction: \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).