Solution 2.2 $D = orth(D_0)$ and $D_\perp = null(D_0^*)$

$$D = \begin{pmatrix} 0.4287 & 0.8060 \\ 0.5663 & 0.1124 \\ 0.7039 & -0.5812 \end{pmatrix}, \quad D_\perp = \begin{pmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{pmatrix}$$

Solution 2.10

- Note that for any $v \in \mathbb{C}^k$, we have

$$\|v\|_\infty \leq \|v\|_2 \leq \sqrt{k}\,\|v\|_\infty$$

These inequalities imply that for any $x \in \mathbb{C}^n$:

$$\frac{1}{\sqrt{m}}\,\|Ax\|_2 \leq \|Ax\|_\infty \leq \|Ax\|_2$$

$$\implies \frac{1}{\sqrt{m}}\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{1}{\sqrt{m}}\frac{\|Ax\|_2}{\|x\|_\infty} \leq \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \frac{\|Ax\|_2}{\|x\|_\infty} \leq \frac{\|Ax\|_2}{\frac{1}{\sqrt{n}}\,\|x\|_2} = \sqrt{n}\frac{\|Ax\|_2}{\|x\|_2}$$

$$\implies \frac{1}{\sqrt{m}}\,\|A\|_2 \leq \|A\|_\infty \leq \sqrt{n}\,\|A\|_2$$

- Similarly, for any $v \in \mathbb{C}^k$, we have

$$\|v\|_2 \leq \|v\|_1 \leq \sqrt{k}\,\|v\|_2$$

These inequalities imply that for any $x \in \mathbb{C}^n$:

$$\|Ax\|_2 \leq \|Ax\|_1 \leq \sqrt{m}\,\|Ax\|_2$$

$$\implies \frac{1}{\sqrt{n}}\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\|Ax\|_2}{\|x\|_1} \leq \frac{\|Ax\|_1}{\|x\|_1} \leq \sqrt{m}\frac{\|Ax\|_2}{\|x\|_1} \leq \sqrt{m}\frac{\|Ax\|_2}{\|x\|_2}$$

$$\implies \frac{1}{\sqrt{n}}\,\|A\|_2 \leq \|A\|_1 \leq \sqrt{m}\,\|A\|_2$$

- Finally, for any $v \in \mathbb{C}^k$, we have

$$\|v\|_\infty \leq \|v\|_1 \leq k\,\|v\|_\infty$$

These inequalities imply that for any $x \in \mathbb{C}^n$:

$$\|Ax\|_\infty \leq \|Ax\|_1 \leq m\,\|Ax\|_\infty$$

$$\implies \frac{1}{n}\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \frac{\|Ax\|_\infty}{\|x\|_1} \leq \frac{\|Ax\|_1}{\|x\|_1} \leq m\frac{\|Ax\|_\infty}{\|x\|_1} \leq m\frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$\implies \frac{1}{n}\,\|A\|_\infty \leq \|A\|_1 \leq m\,\|A\|_\infty$$

Solution 2.14 Without loss of generality, assume

$$X_{22} = \begin{bmatrix} U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{21}^* \\ U_{22}^* \end{bmatrix}$$

with $\Sigma_1 = \Sigma_1^* > 0$ and $U_2 = \begin{bmatrix} U_{21} & U_{22} \end{bmatrix}$ unitary. Then

$$\text{Ker } X_{22} = \text{span}\{\text{columns of } U_{22}\}$$

and

$$X_{22}^+ = \begin{bmatrix} U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{21}^* \\ U_{22}^* \end{bmatrix}.$$

Moreover

$$\begin{bmatrix} I & 0 \\ 0 & U_2^* \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12}U_{21} & X_{12}U_{22} \\ U_{21}^*X_{12}^* & \Sigma_1 & 0 \\ U_{22}^*X_{12}^* & 0 & 0 \end{bmatrix} \geq 0$$

gives $X_{12}U_{22} = 0$. Hence, Ker $X_{22} \subset$ Ker X_{12} and now

$$X_{12}X_{22}^+X_{22} = X_{12}U_{21}U_{21}^* = X_{12}U_{21}U_{21}^* + X_{12}U_{22}U_{22}^* = X_{12}.$$

The factorization follows easily.