Solution 7.4 The Hankel singular values are listed in the following table:

	σ_1	σ_2	σ_3	σ_4	σ_5
$\alpha = 2$	1.572	0.641	0.209	0.061	0.017
$\alpha = 4$	0.992	0.689	0.418	0.245	0.156
$\alpha = 20$	0.589	0.549	0.497	0.449	0.416
$\alpha = 100$	0.517	0.510	0.500	0.500	0.483

Hence $\sigma_i \to 0.5$ as $\alpha \to \infty$.

Solution 7.5 Let T_1 be the balancing transformation such that $T_1 P T_1^* = \Sigma$, $(T_1^*)^{-1} Q T_1^{-1} = \Sigma$. Then output normalized realization is obtained by $T = \Sigma^{1/2} T_1$.

Solution 7.7 A state space realization of $G_d(z)$ is given by

$$G_d(z) = \left[\begin{array}{c|c} -(I-A)^{-1}(I+A) & -\sqrt{2}(I-A)^{-1}B \\ \hline \sqrt{2}C(I-A)^{-1} & C(I-A)^{-1}B+D \end{array}\right] =: \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$$

Then it can be verified that

$$A_d P_d A_d^* - P_d + B_d B_d^* = 0, \quad A_d^* Q_d A_d - Q_d + C_d^* C_d = 0$$

are satisfied by $P_d = P$ and $Q_d = Q$ where

$$AP + PA^* + BB^* = 0, \quad A^*Q + QA + C^*C = 0.$$

Solution 7.8 Use MATLAB:

- $T =??$ and $n =???$
- $g = nd2sys([-0.05, 1], [0.05, 1])$; g5=mmult(g,g,g,g,g);
- g5=sysbal(g5); % to avoid numerical problem.
- $g10 = mmult(g5, g5); gt = nd2sys(1, [T, 1])$; sys=mmult(g10,gt);
- $[sysb, sig] = sysbal(sys); sysr = strunc(sysb, n)$;
- err=msub(sysb,sysr); e=hinfnorm(err,0.0000001)

For $T = 0$, all Hankel singular values are 1. Thus no reduced order model can be found.

For $T = 0.01$, the Hankel singular values are

$$\sigma_i = 0.99953,\ 0.99803,\ 0.99521,\ 0.99047,\ 0.98259,\ 0.96907$$
$$0.94442;\ 0.89583,\ 0.79362,\ 0.58249,\ 0.22052.$$

Thus no reduced order model can be found.

For $T = 0.1$, the Hankel singular values are

$$\sigma_i = 0.97158,\ 0.89465,\ 0.78849,\ 0.67228,\ 0.55857,\ 0.45297$$
$$0.35669,\ 0.26874,\ 0.18736,\ 0.11060,\ 0.03657.$$

A tenth order approximation gives

$$\|G - G_{10}\|_\infty = 0.07313, \quad bound = 0.07313.$$

For $T = 1$, the Hankel singular values are

$$\sigma_i = 0.73719,\ 0.35146,\ 0.18249,\ 0.114322,\ 0.07921,\ 0.05744$$
$$0.04212,\ 0.03031,\ 0.02052,\ 0.01191,\ 0.00391.$$

A fifth order approximation gives

$$\|G - G_5\|_\infty = 0.07391, \quad bound = 0.33243.$$

For $T = 10$, the Hankel singular values are

$$\sigma_i = 0.54386,\ 0.05715,\ 0.02054,\ 0.01201,\ 0.00814,\ 0.00584$$
$$0.00426,\ 0.00306,\ 0.00207,\ 0.00120,\ 0.00039.$$

A first order approximation gives

$$\|G - G_1\|_\infty = 0.08792, \quad bound = 0.22935.$$