Multivariable Control

Lecture 05
Multivariable Poles and Zeros

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SISO transfer function: $G(s) = \frac{n(s)}{d(s)}$ (no common factors between $n(s)$ and $d(s)$).

Poles: values of $s$ such that $d(s) = 0$.

Zeros: values of $s$ such that $n(s) = 0$.

Watch out for pole/zero cancellation in cascaded systems!

(Find the state space realization.)
MIMO Case

For MIMO: \( G(s) = D + C(sI - A)^{-1}B \) (rational matrix: matrix of rational functions). We will write \( G(s) \) as

\[
G(s) = D_L^{-1}(s)N_L(s) = N_R(s)D_R^{-1}(s)
\]

where \( D_L, N_L, N_R, D_R \) are polynomial matrices.
Some Definitions

Definition
\[ \mathbb{R}^{p \times m}[s] = p \times m \text{ polynomial matrix of } s \text{ with coefficients in } \mathbb{R} \]
\[ \mathbb{R}^{p \times m}(s) = p \times m \text{ rational matrix of } s \text{ with coefficients in } \mathbb{R} \]
\[ \mathbb{R}_p(s) = \text{ proper rational functions} \]
\[ \mathbb{R}_{p,o}(s) = \text{ strictly proper rational functions} \]

Definition
The normal rank of \( G \in \mathbb{R}^{p \times m}[s] \) is the maximum rank of \( G \) over all \( s \in \mathbb{C} \).

Definition (Generalization of the concept of scalar constants in \( \mathbb{R}[s] \))
\( G \in \mathbb{R}^{k \times k}[s] \) is unimodular if \( G^{-1} \in \mathbb{R}^{k \times k}[s] \).

Fact: \( G \) is unimodular iff \( G^{-1} \in \mathbb{R}[s] \) iff \( \det G = \text{constant} \).
\[
\begin{bmatrix}
1 & s + 1 \\
0 & 2
\end{bmatrix}
\]
is unimodular, but
\[
\begin{bmatrix}
1 & 2 \\
0 & s + 1
\end{bmatrix}
\]
is not.
Elementary Operations

- row/column
  - interchange
  - scaling
  - scaling and addition

row: pre-multiplication
column: post-multiplication

All elementary operations are unimodular.
Let $M \in \mathbb{R}^{p \times m}[s]$ with $\text{rank}(M) \leq \min(p, m)$. Then $M(s)$ can be transformed using elementary operations to

$$U(s)M(s)V(s) = S(s) = \begin{bmatrix} \text{diag}\{\varepsilon_1(s), \ldots, \varepsilon_r(s)\} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\varepsilon_i(s)$ is monic and $\varepsilon_i | \varepsilon_{i+1}$ ($\varepsilon_i$ divides $\varepsilon_{i+1}$). \{\varepsilon_i\} are called the invariant factors of $M(s)$. 
Let $G \in \mathbb{R}^{p \times m}(s)$. $G$ can be written as $G(s) = \frac{N(s)}{d(s)}$ where $N \in \mathbb{R}^{p \times m}[s]$ and $d \in \mathbb{R}[s]$ (l.c.m. of all denominators of $G_{ij}(s)$). Reduce $N(s)$ to its Smith Form, then

$$U(s)G(s)V(s) = U(s)N(s)V(s)/d(s) = S(s)/d(s).$$

Reduce $\sigma_i(s)/d(s)$ to lowest terms:

$$\frac{\sigma_i(s)}{d(s)} = \frac{\varepsilon_i(s)}{\psi_i(s)}.$$

Then

$$U(s)G(s)V(s) = \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\varepsilon_r(s)}{\psi_r(s)} \end{bmatrix}.$$
MIMO poles/zeros

zero polynomial: \( z(s) = \varepsilon_1(s) \cdots \varepsilon_r(s) \). Roots are called transmission zeros of \( G(s) \)
pole polynomial: \( p(s) = \psi_1(s) \cdots \psi_r(s) \). Roots are called poles of \( G(s) \). \( \partial p \) (order of \( p(s) \)) is called the McMillan degree of \( G(s) \) (i.e., number of poles).

If \( G \) is square:
- transmission zeros are also the roots of \( \det G(s) = 0 \) (but must be careful with pole/zero cancellation).
- if, in addition, \( G(\infty) \) is invertible, poles of \( G^{-1}(s) \) are the transmission zeros of \( G(s) \) (again watch out for pole/zero cancellation).
Example

Consider

\[
G(s) = \begin{bmatrix}
\frac{s+1}{s+3} & \frac{s-1}{s+4} \\
0 & \frac{1}{(s+1)(s+3)}
\end{bmatrix}
\]

Smith-McMillan Form:

\[
G_{SM}(s) = \begin{bmatrix}
\frac{1}{(s+1)(s+3)(s+4)} & 0 \\
0 & \frac{(s+1)(s+4)}{(s+3)}
\end{bmatrix}
\]

Clearly, there is no pole/zero cancellation for the $-1$ and $-4$ modes since these poles and zeros occur in different channels.
Write the Smith-McMillan Form of $G(s)$ as

$$U(s)G(s)V(s) = E(s)\Psi^{-1}_r(s) = \Psi^{-1}_\ell(s)E(s).$$

Then $G(s)$ can be written as

$$G(s) = \underbrace{(U^{-1}(s)E(s))}_{N_r(s)}\underbrace{(V(s)\Psi_r(s))^{-1}}_{D_r(s)} = \underbrace{(\Psi_\ell(s)U(s))^{-1}}_{D_\ell(s)}\underbrace{(E(s)V^{-1}(s))}_{N_\ell(s)}.$$
MIMO transfer matrix: \( G(s) = N_r(s)D_r^{-1}(s) = D_\ell^{-1}(s)N_\ell(s) \) where \( N_r, D_r, N_\ell, D_\ell \) are polynomial matrices (matrix fraction description, MFD). \( D_r \) is \( m \times m \), \( D_\ell \) is \( p \times p \), \( N_r \) is \( p \times m \), \( N_\ell \) is \( m \times p \).

Coprime MFD:

\((N_r, D_r)\) are right coprime: If there are polynomial matrices \( N_{r1}, D_{r1}, R \) such that

\[
N_r = N_{r1}R \quad , \quad D_r = D_{r1}R
\]

(so that \( N_rD_r^{-1} = N_{r1}D_{r1}^{-1} \)), then \( R \) must be unimodular, i.e., \( R \) and \( R^{-1} \) are both polynomial matrices (like nonzero constants for SISO systems).

\((D_\ell, N_\ell)\) are left coprime: If there are polynomial matrices \( D_{\ell1}, N_{\ell1}, L \) such that

\[
D_\ell = LD_{\ell1} \quad , \quad N_\ell = LN_{\ell1}
\]

(so that \( D_\ell^{-1}N_\ell = D_{\ell1}^{-1}N_{\ell1} \)), then \( L \) must be unimodular.
Condition for Coprimeness

Row reduce \[ \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} \] to \[ \begin{bmatrix} R(s) \\ 0 \end{bmatrix} \] (i.e., find an unimodular matrix \( X \) such that \( X(s) \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix} \)). Then \( (N_r, D_r) \) is right coprime if and only if \( R(s) \) is unimodular.

Column reduce \[ \begin{bmatrix} D_\ell(s) & N_\ell(s) \end{bmatrix} \] to \[ \begin{bmatrix} L(s) & 0 \end{bmatrix} \] (i.e., find an unimodular matrix \( Y \) such that \( \begin{bmatrix} D_\ell(s) & N_\ell(s) \end{bmatrix} Y(s) = \begin{bmatrix} L(s) & 0 \end{bmatrix} \)). Then \( (D_\ell, N_\ell) \) is left coprime if and only if \( L(s) \) is unimodular.
Row and Column Reductions

To row reduce $\begin{bmatrix} N_r \\ D_r \end{bmatrix}$, first find its Smith Form $U \begin{bmatrix} N_r \\ D_r \end{bmatrix} V = \Sigma$, then

$$U \begin{bmatrix} N_r \\ D_r \end{bmatrix} = \Sigma V^{-1} = \begin{bmatrix} R \\ 0 \end{bmatrix}.$$ 

Similarly for column reduction.

MATLAB Tools

Use `smith` command in MAPLE (in MATLAB Symbolic toolbox) to transform

$$\begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix}$$

or $\begin{bmatrix} D_\ell(s) & N_\ell(s) \end{bmatrix}$ to the Smith Form.
Bezout Identity

If \((N_r, D_r)\) is right coprime, then there exist polynomial matrices \((U_r, V_r)\) such that

\[ U_r N_r + V_r D_r = I. \]

If \((D_\ell, N_\ell)\) is left coprime, then there exist polynomial matrices \((V_\ell, U_\ell)\) such that

\[ D_\ell V_\ell + N_\ell U_\ell = I. \]

Generalized Bezout Identity

If \((N_r, D_r)\) are right coprime and \((D_\ell, N_\ell)\) are left coprime, then there exist polynomial matrices \((U_r, V_r, U_\ell, V_\ell)\) such that

\[
\begin{bmatrix}
U_r & V_r \\
D_\ell & -N_\ell
\end{bmatrix}
\begin{bmatrix}
N_r & V_\ell \\
D_r & -U_\ell
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

\((U_r, V_r, U_\ell, V_\ell)\) may be found using \texttt{smith} function in MATLAB symbolic toolbox.
Computation of Coprime MFD

Given a right MFD for \( G(s) = N_{r_1}(s)D_{r_1}^{-1}(s) \) (e.g., \( G(s) = \frac{N(s)}{d(s)} \)) where \( d(s) \) is the least common multiple of all denominators of \( G(s) \)). Extract the greatest common right divisor, \( R \), for \((N_{r_1}, D_{r_1})\). Then \( N_{r}(s) = N_{r_1}(s)R^{-1}(s) \) and \( D_{r}(s) = D_{r_1}(s)R^{-1}(s) \) form a right coprime MFD for \( G(s) \).

Similarly, to find the left coprime MFD, start with any left MFD \((D_{\ell_1}, N_{\ell_1})\). Extract the greatest common left divisor, \( L \). Then \( D_{\ell} = L^{-1}D_{\ell_1} \) and \( N_{\ell} = L^{-1}N_{\ell_1} \) form a left coprime MFD.

Coprime MFD can be used in pole/zero characterization, state space realization and polynomial based control design.

Example: \( G_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} \) and \( G_2(s) = \begin{bmatrix} \frac{s+1}{s+3} & \frac{s-1}{s+4} \\ 0 & \frac{1}{(s+1)(s+3)} \end{bmatrix} \).
MIMO Pole/Zero: Characterization through MFD

Given coprime MFD’s for $G(s): (N_r(s), D_r(s))$ and $(D_\ell(s), N_\ell(s))$.

Poles: values of $s$ such that $\det D_r(s) = \det D_\ell(s) = 0$.

If $(A, B, C, D)$ is a minimal realization of $G(s)$ (state dimension, $n = \#$ of poles), then poles are also given by the roots of $\det(sI - A) = 0$ (characteristic equation), i.e., eigenvalues of $A$. (Note that $G(s) = C \text{Adj}(sI - A)B/\det(sI - A) + D$.)

Transmission Zeros: values of $s$ such that $N_r(s)$ (or $N_\ell(s)$) loses rank.

Blocking zeros: values of $s$ such that $G(s) = 0$.

Blocking zeros $\subset$ Transmission zeros

Pole/zero cancellation in MIMO system may not be obvious, e.g., consider $K(s)G(s)$ below

$$G(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{1}{s+2} \\
\frac{2}{s+2} & \frac{1}{s+1}
\end{bmatrix}, \quad K(s) = \begin{bmatrix}
\frac{s+2}{s-\sqrt{2}} & -\frac{s+1}{s-\sqrt{2}} \\
0 & 1
\end{bmatrix}.$$
Interpretation of poles and zeros

If $s_o$ is a pole of $G(s)$, then there exists an initial condition $x_o$ such that the zero input response of $G$ is $e^{s_o t}C x_o$ (choose $x_o$ to be the eigenvector of $A$ corresponding to $s_o$).

If $z_o$ is a transmission zero and $p \geq m$ (i.e., $G$ is square or tall), then there exist $u_o$ and $x_o$ such that the output under the input $u(t) = u_o e^{z_o t}$ and initial state $x_o$ is zero (certain input direction is blocked).

If $z_o$ is a transmission zero and $p \leq m$ (i.e., $G$ is square or fat), then there exist $v_o$ and $x_o$ such that under any input of the form $u_o e^{z_o t}$ and initial state $x_o$, output $y(t)$ is orthogonal to $v_o$, i.e., $v_o^T y(t) = 0$ (certain output direction cannot be achieved with any input).

If $z_o$ is a blocking zero, then $G(s) u_o e^{z_o t} = 0$ for all $u_o$.

MATLAB commands: pole, tzero
Example

\[
G(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{1}{s+2} \\
\frac{2}{s+2} & \frac{1}{s+1}
\end{bmatrix}
\]

poles: \{-1,-1,-2,-2\}  \quad \text{zero: -1.5}

Evaluate \(G\) at the transmission zero:

\[
G(-1.5) = \begin{bmatrix}
-2 & 2 \\
2 & -2
\end{bmatrix}
\]

zero input direction: \(u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

unreachable output direction: \(v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).