Correction of Singularity Computation for Iterative Control of Nonlinear Affine Systems

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Abstract
The ball-on-plate example in the paper by the authors [1] contains a mistake in the vector field. This note presents the singularity result of the corrected vector field.

1 Singularity for Ball-on-Plate Example
The kinematics of the ball rolling on a flat plate is given by [2]:

\[ \dot{X} = \begin{bmatrix} g_1 & g_2 \end{bmatrix} U \]  \hspace{1cm} (1)

where

\[ X = \begin{bmatrix} x \\ y \\ u \\ v \\ \phi \end{bmatrix} \quad U = \begin{bmatrix} -w_y \\ wx \end{bmatrix} \]

\((x, y)\) is the position coordinate of the center of the ball, \((u, v)\) are the rotational angles of with respect to the \(x\) and \(y\) axes, \(\phi\) is the rotation angle about the \(z\) axis, \((w_x, w_y)\) are the linear velocities of the ball in the \(x\) and \(y\) directions. The vector fields, \((g_1, g_2)\), is given by

\[ g_1 = \begin{bmatrix} 0 \\ -1 \frac{\sin(\phi)}{\cos(v)} \\ \cos(\phi) \\ \tan(v) \sin(\phi) \end{bmatrix} \quad g_2 = \begin{bmatrix} 1 \\ 0 \frac{\cos(\phi)}{\cos(v)} \\ -\sin(\phi) \\ \tan(v) \cos(\phi) \end{bmatrix} \]

Through the following coordinate transformation

\[ \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \begin{bmatrix} \cos(v) \sin(\phi) & \cos(\phi) \\ \cos(v) \cos(\phi) - \sin(\phi) & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \]

and reordering of the state to

\[ Z = \begin{bmatrix} u \\ v \\ \phi \\ x \\ y \end{bmatrix} \]

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we obtain the more familiar form:

\[
\dot{Z} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}
\]  

(2)

where

\[
f_1 = \begin{bmatrix} 1 \\ 0 \\ \sin(v) \\ \cos(\phi) \cos(v) \\ -\sin(\phi) \cos(v) \\ -\cos(\phi) \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sin(\phi) \\ -\cos(\phi) \end{bmatrix}.
\]

(3)

We can now apply Lemma 2 in [1] to identify the singular controls. Instead of the vector fields \((f_1, f_2)\) we use the original vector fields \((g_1, g_2)\) to compute the Grammian:

\[
Q_c(t) = [B(t), \Delta B(t), \cdots, \Delta^{n-1} B(t)]
\]

which is a \(5 \times 10\) matrix. If \(c_i\) is the \(i\)-th column of \(Q_c\), let \(Q_1\) be the square submatrix consisting of columns \((c_1, c_2, c_3, c_4, c_5)\). Using the symbolic toolbox in MATLAB, we compute

\[
\det(Q_1) = \frac{(U_2)^3}{\cos(v)} (U_2 \dot{U}_1 - U_1 \dot{U}_2).
\]

Therefore if \(U_2 \dot{U}_1 - U_1 \dot{U}_2\) is not identically zero on \([0, T]\), then the controllability Grammian is nonsingular. To verify that this singularity condition is also sufficient, we compute the determinants of all \(5 \times 5\) submatrices and substitute in the singularity condition

\[
U_2 \dot{U}_1 = U_1 \dot{U}_2.
\]

(4)

All the determinants are found to be zero (the equivalent condition \(\det(Q_c Q_c^T) = 0\) leads to an expression too large for the symbolic toolbox).

Note that this singularity condition is the same result as in the linearize ball-on-a-plate example, and reduces to

\[
w_x(t) = Cw_y(t), \ \forall t \in [0, T],
\]

where \(C\) is a constant.

This singularity result also includes the singularity found in [3, 4] which corresponds to either \(w_x(t) = 0\) or \(w_y(t) = 0\) for all \(t \in [0, T]\) (though the result was stated in terms of \(W_1(t)\) and \(W_2(t)\)).

2 Conclusion

The vector field for the ball-on-plate example in [1] contains an error. This paper shows that the singularity condition actually coincides with the linearized ball-on-plate system.

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References


