Finite dimensional controller design for infinite dimensional systems: The circle criterion approach

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Abstract: The Hilbert space generalization of the circle criterion is used for the finite dimensional compensator design for unstable infinite dimensional systems. A design algorithm which generalizes the small gain based algorithm in [5] is presented.

Keywords: Passivity method; distributed parameter systems; reduced order controller design.

1. Introduction

Many physical processes are naturally modeled as linear time invariant distributed parameter systems. To obtain the specified performance, often external feedback compensation is needed. In some cases, the performance requirement can be translated into a stability requirement by modifying the open loop (uncontrolled) process. It is the aim of this paper to address such stabilization problems with the finite dimensionality (i.e. implementability) constraint on the compensator.

There are two possible approaches to the stabilization problem: Find a finite dimensional approximation of an infinite dimensional controller (the indirect approach) or find a controller for a finite dimensional approximation of the open loop system (the direct approach). The ultimate question is: Does the finite dimensional controller stabilize the infinite dimensional system? It has been shown by examples that the control action and the unmodeled dynamics (due to approximations to the controller or the plant) can interact (the so-called spill-over effect) in an adverse manner to cause instability [1]. Many results have appeared in the literature to address this issue. A wide range of methodologies exist, but for most, the granularity of approximation cannot be determined a priori. A sequence of approximation, design, and stability validation have to be performed; these methods guarantee the convergence of this process but does not provide information on the rate of convergence.

To describe the situation conceptually, imagine the true system (either the plant, \( P_o \), or the controller, \( C_o \), depending on whether the direct or the indirect approach has been used) and its approximate (\( P_N \) or \( C_N \)) lying in some parameter space. The closed loop system for the approximate is stable by design, and, in general, it will have a 'robustness neighborhood' around the nominal design parameters. Most of the methods proposed for this problem guarantee some minimum size of this neighborhood for all \( N \). Thus, for \( N \) sufficiently large, the true system will be inside the robustness neighborhood and is stabilized. A notable exception is [5] where the small gain stability criterion is used to derive an analytic expression for the size of the robustness neighborhood based on the unstable part of the open loop plant only. This paper presents a generalization of [5] by using the Hilbert space version of the circle criterion to estimate the size of the robustness neighborhood instead of the small gain theorem. The result in [5] is included as a special case. The basic idea of our approach can be described by the following steps:

1. Decompose the open loop system, \( P_o \), into the sum of an unstable portion, \( P_u \), a finite dimensional exponentially stable portion, \( P_s \), that approximates the stable part of the plant, and an infinite dimensional exponentially stable residue, \( P_r \).
(2) With any feedback controller \( K \) that stabilizes \( P_u + P_s \), the closed loop system can be written as a feedback interconnection between a stable forward system involving \( K \) and \( P_u + P_s \) and a feedback which is the residue \( P_r \). The forward block can be further transformed into a system involving only \( P_u \) and a compensator that stabilizes it.

(3) From the circle criterion, the smallest circle that encloses the generalized Nyquist diagram (will be defined in Section 2) of the forward system determines the maximum allowable size of the feedback system \( P_r \); this size is stated in terms of the smallest circle that encloses the generalized Nyquist diagram of \( P_r \). Since the forward system only depends on \( P_u \) and its stabilizing compensator, so does the required level of approximation (i.e., the size of \( P_r \)).

A natural optimization problem is then to find a compensator that (1) stabilizes \( P_u \) and (2) maximizes the radius of the circle for \( P_r \). This problem is an \( H_{\infty} \) optimization problem [7] which can be transformed to the standard Nehari problem. An analytic bound can then be obtained. It turns out that the least conservative form of this bound is the same as the \( H_{\infty} \) bound. This procedure is then modified by moving all the unstable zeros in the forward system to the feedback system. Through an example, we show that a less conservative bound on \( P_r \) is obtained. This approach is a generalization of the small gain method in [5] since the small gain bound is obtained when the circles used in the stability analysis are restricted to center around the origin in the corresponding Nyquist diagrams.

The paper is organized as follows. Section 2 discusses the concept of passivity viewed from the input/output spaces, the frequency domain, and the state space. The Hilbert space version of the hyperstability theorem is then stated, with the circle criterion as a natural extension. Details have been omitted in this section, since a more elaborate presentation can be found in [19]. Section 3 presents the design procedure for the stabilizing controller, leading to two design algorithms. An example of a scalar system with a single unstable mode is also studied here.

The usual notations of \( \geq 0 \) and \( > 0 \) are used to denote positive semidefiniteness and positive definiteness of matrices, respectively. The symbols \( \lambda_{\min}(A) \), \( \mu_{\min}(A) \) and \( \sigma_{\min}(A) \) are defined as the minimum matrix eigenvalue, minimum eigenvalue of \( \frac{1}{2}(A + A^*) \) (* denotes the complex conjugate transpose) and minimum matrix singular value, respectively. A coercive operator means a positive operator that is also bounded invertible in the space under consideration. The notation \( \gg 0 \) is used to denote coercive operators. The space in which norms and inner products are taken will not be noted explicitly; the interpretation is inferred from the arguments. The truncated \( L_2 \) space, \( L_2(\mathbb{R}, t) \), is denoted by \( L_2t \). The inner product and norm in \( L_2 \) is denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \|_2 \), respectively. We say \( x \in L_2t \) for all \( t \in [0, \infty) \). The space of bounded linear operators from a Hilbert space \( X \) to a Hilbert space \( Y \) is denoted by \( \mathcal{L}(X, Y) \), and \( \mathcal{L}(X) \triangleq \mathcal{L}(X, X) \).

2. Passivity, positive realness and hyperstability in Hilbert spaces

For a given linear time invariant distributed parameter system, this section gives a sufficient condition for the state space exponential stability in terms of dissipativeness of its subsystems. This condition is called the circle criterion due to its graphical interpretation in the Nyquist diagram. It is the basis of the design algorithm in the next section.

The system under consideration is modeled by a linear abstract evolution equation on a real Hilbert space \( \mathcal{X} \):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathcal{X}, \\
y(t) &= Cx(t) + Du(t).
\end{align*}
\] (2.1a, 2.1b)

The operator \( A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X} \) is the infinitesimal generator of a \( C_0 \)-semigroup, \( U(t) \). The operators, \( B : \mathbb{R}^m \to \mathcal{X}, \ C : \mathcal{X} \to \mathbb{R}^n, \ D : \mathbb{R}^m \to \mathbb{R}^n \) are all assumed bounded.

The solution of (2.1), \( x(t) \), is considered in the mild sense [15]:

\[
x(t) = U(t)x_0 + \int_0^t U(t-s)Bu(s) \, ds.
\] (2.2)
The strong differentiability of $x(t)$ is not imposed. The existence and uniqueness of $x(t)$ under nonlinear feedback interconnections have been considered in [19]; these issues will not be further discussed here. A $C_0$-semigroup $U(t)$ is said to be exponentially stable if there exists $M \geq 1$ and $\sigma > 0$ such that
\[
\|U(t)\| \leq M e^{-\sigma t}.
\] (2.3)
The system described by (2.1) is said to be exponentially stable if $A$ generates an exponentially stable $C_0$-semigroup.

We will consider the passivity of (2.1) from several view points: input/output spaces, frequency domain, and state space. Following the definition in [21,10] (in a slightly restrictive form), we define passivity as below:

**Definition 1.** Let $H : L_{2e}(\mathbb{R}_+ ; \mathbb{R}^m) \to L_{2e}(\mathbb{R}_+ ; \mathbb{R}^m)$ be a general dynamical system. We say that $H$ is $(\gamma, \xi)$-passive if
\[
\int_0^t e^{\gamma s} (Hu)(s)^T u(s) \, ds \geq \xi \int_0^t e^{\xi s} \|u(s)\|^2 \, ds.
\] (2.4)
for all $u \in L_2(0, t)$.

The inequality (2.4) can be written in terms of inner products on $L_2(0, t)$ defined in terms of exponential weighted integrals [6, Chapter VI]. When $\xi = 0$, the above is the usual passivity definition (also called the Popov inequality [13]). When $\xi > 0$, the system is sometimes called $u$-strictly-passive [9], or simply, strictly passive [6, Definition VI.4.2].

The definition for passivity is motivated in part by the energy dissipativeness of a network. Consider a multi-port network with the voltage as the input and the corresponding current as the output. The total (exponentially weighted) energy delivered to the network from time 0 to $t$ is the left hand side quantity in (2.4). If the network is initially relaxed and $\gamma \geq 0$, then energy is always delivered to the system; hence, the network either conserve or dissipates energy. When $\xi > 0$, then the rate of energy dissipation is greater than some minimal rate determined by the input.

If a system is defined by a convolution kernel [6, Appendix C], then passivity can be equivalently stated in the frequency domain (via the Fourier transformation). Follow the definition in [2,11], we first introduce the convolution algebra:

**Definition 2.** For $\sigma_0 \in \mathbb{R}$, $f \in \mathcal{A}(\sigma_0)$ if
\[
f(t) = \begin{cases} 0 & \text{for } t < 0, \\ f_0(t) + f_{sp}(t) & \text{for } t \geq 0,
\end{cases}
\]
and there exists $\sigma < \sigma_0$ such that (i) $e^{-\sigma t} f_0 \in L_1(\mathbb{R}_+)$, and (ii) $f_{sp}(t) = \sum_{i=0}^{\infty} f_i \delta(t - t_i)$, where $\delta(\cdot)$ denotes the Dirac delta, $f_i \in \mathbb{R}$, for all $i = 0, 1, 2, \cdots$, and $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$.

We now consider convolution systems with the convolution kernel being a matrix with each element in $\mathcal{A}(\sigma_0)$ for some $\sigma_0 \leq 0$ (denoted by $M(\mathcal{A}(\sigma_0))$). By using Parseval's Theorem, the following fact can be simply shown [6, Example VI.4.1]:

**Fact 1.** Let $H : L_{2e}(\mathbb{R}_+ ; \mathbb{R}^m) \to L_{2e}(\mathbb{R}_+ ; \mathbb{R}^m)$ be defined by
\[
Hu = h \ast u \quad \text{where } h \in M(\mathcal{A}(\sigma_0)) \text{ and } u \in L_{2e}.
\]
Then $H$ is $(2\gamma, \xi)$-passive if and only if
\[
\mu_{\min}(\hat{h}(\omega - \gamma)) \geq \xi
\] (2.5)
where $\hat{h}$ is the elementwise Fourier transform of $h$. 
Note that $\hat{h}(j\omega - \gamma)$ is well defined since $h \in M(A(-\gamma))$. If $H$ has a state space realization as given by (2.1), then it is easy to show [11] that its convolution kernel belongs to $M(A(-\gamma))$ for some $\gamma \in \mathbb{R}$. If $A$ generates an exponentially stable $C_0$-semigroup, then $\gamma > 0$. By using the frequency condition, it is easy to show that $(0, \xi)$-passivity for some $\xi > 0$ implies $(\gamma, 0)$-passivity for some $\gamma > 0$ which in turn implies passivity.

So far passivity is introduced as an input-output property. When there is an underlying state space representation, then passivity can be stated as a set of algebraic equations. In finite dimensions, this fact is the well-known Kalman–Yakubovich Lemma [12,23]. The generalization to the abstract evolution system described by (2.1) was first given in [20,14]. This result is stated here in the same form as in [19].

**Fact 2.** Let $H : L_{2e}(\mathbb{R}_+ ; \mathbb{R}^n) \to L_{2e}(\mathbb{R}_+ ; \mathbb{R}^n)$ be defined by $y = H(u)$ and (2.1). Suppose the $C_0$-semigroup associated with $H$ satisfies (2.3). Then for $\gamma < 2\sigma$, $\xi < \sigma_{\min}(D)$, $H$ is $(\gamma, \xi)$-passive if and only if for each $\xi_0 < \xi$, there exist $P$, $L$, both in $\mathcal{L}(X)$, $P > 0$, $L > 0$, and $Q \in \mathcal{L}(X, \mathbb{R}^m)$, $W \in \mathbb{R}^{m \times m}$, such that

\[
\begin{align*}
(A^*P + PA + 2\gamma P + L + Q^*Q)x &= 0 \quad \text{for all } x \in \mathcal{D}(A), \\
B^*P &= C - W^*Q, \\
W^*W &= D + D^* - 2\xi_0 I.
\end{align*}
\]  

(2.6a) (2.6b) (2.6c)

From the various conditions on passivity, it is clear that a constant feedforward (directly from $u$ to $y$) can be added to any linear time invariant system to render it passive. The least amount of such feedforward is a quantity that we shall use later. We have the following definitions [19].

**Definition 3.** (1) Let $H$ be given by (2.1) and exponentially stable. The $\nu$-index is defined as

\[\nu(H) = \inf \{ \lambda \in \mathbb{R} : (A, B, C, D) \text{ satisfies (2.6) with } \xi_0 = -\lambda \text{ and } \gamma = 0 \} .\]

(2) Let $H$ be a linear time invariant system with kernel $h \in M(A(-\gamma))$ for some $\gamma > 0$. The $\nu_F$-index is defined as

\[\nu_F(H) = -\inf_{\omega \in \mathbb{R}} \mu_{\min}(\hat{h}(j\omega)) .\]

It can be shown [19] that $\nu_F(H) = \nu(H)$, so that the state space definition can be used for the stability analysis and the frequency domain definition for the computation. It is also clear that $\nu(H) \leq 0$ if and only if $H$ is passive, and that $\nu(H) < 0$ implies that $H$ is strictly passive.

Given an exponentially stable linear time invariant system $H$ described by (2.1), it is called $\gamma$-hyperstable if $e^{\gamma t}x(t) \to 0$ as $t \to \infty$ when the input $u$ is given by $u = G(y)$ where $G : L_{2e}(\mathbb{R}_+ ; \mathbb{R}^n) \to L_{2e}(\mathbb{R}_+ ; \mathbb{R}^n)$ is $(\gamma, 0)$-passive. The Hilbert space generalization of the Hyperstability Theorem [16] can be concisely stated in terms of the $\nu$-index.

**Theorem 1.** If $\nu(H) < 0$, then $H$ is $\gamma$-hyperstable for some $\gamma > 0$.

If the feedback system $G$ is also linear time invariant with kernel $g$ and a state space representation such as (2.1), and it is exponentially stable, when $\nu(G) < 0$ implies that $G$ is $(\gamma, 0)$-passive for some $\gamma > 0$ and the state of $H$ tends to zero exponentially with rate $\gamma$. Furthermore, by reversing the roles of $H$ and $G$ in Theorem 1, it is clear that the internal state of $G$ will also tend to zero exponentially as $t \to \infty$.

For the feedback interconnected system of $H$ and $G$, the condition in Theorem 2 corresponds to a specific ‘tearing’ [22] (partition between feedforward and feedback blocks). More general conditions are possible by using loop transformations (adding a constant loop to both $H$ and $G$) [6]. If a feedforward loop of $1/\beta$ and a feedback loop of $\alpha$ are added to $H$ and the corresponding loops are added to $G$ to cancel the net effect, then an application of Theorem 1 produces the following generalization.
Corollary 1. Define
\[ \tilde{H} = (I - \alpha H)^{-1} H. \]
If there exist real numbers \( \alpha, \beta, \) and a positive number \( \gamma \) such that
\begin{enumerate}
\item \( \tilde{H} \) is internally (in state space) exponentially stable,
\item \( \nu(\tilde{H} + \beta^{-1}I) < 0, \)
\item \( (I - \beta^{-1}(G + \alpha I))^{-1}(G + \alpha I) \) is \((\gamma, 0)\)-passive,
\end{enumerate}
then \( e^{\gamma t}x(t) \rightarrow 0 \) as \( t \rightarrow \infty. \)

Clearly, for \( \alpha = 0 \) and \( \beta = \infty \), Theorem 1 is recovered. In general, if \( G \) is linear time invariant exponentially stable, the conditions in Corollary 1 can be translated to a relationship between \( \alpha, \beta \) and the \( \nu \)-indices of transfer functions involving \( H \) and \( G \). This is summarized below.

(1) Let \( k_H \) be defined as the Nyquist gain of \(-H\) (the maximum constant scalar positive feedback that does not destabilize \( H \)). If \( 0 \leq \alpha < k_H \), then \( \tilde{H} \) is internally exponentially stable. A more conservative condition in terms of the \( \nu \)-index is given by
\[ 0 \leq \alpha < \nu(-H). \tag{2.7} \]

(2) The condition that \( \nu(\tilde{H} + 1/\beta) < 0 \) is clearly equivalent to \( \beta < 1/\nu(\tilde{H}) \). Through further manipulation, the following sufficient condition can be shown [19]:
\[ 1 - (\beta - 2\alpha) \text{Re} \mu_{\min}(\hat{h}(j\omega)) - \alpha(\beta - \alpha) \| \hat{h}(j\omega) \|^2. \tag{2.8} \]

For \( \beta \geq \alpha \geq 0 \), this condition has the interpretation that the generalized Nyquist diagram of \( H \), \( (\| \hat{h}(j\omega) \|^2 - (\text{Re} \mu_{\min}(\hat{h}(j\omega)))^2)^{1/2} \) (with both + signs plotted) vs. \( \text{Re} \mu_{\min}(\hat{h}(j\omega)) \) and denoted by \( \mathcal{N}_H \), is enclosed in a circle symmetric about the \( \text{Re} \mu_{\min}(\hat{h}(j\omega)) \) axis with intersection with the axis at \(-1/(\beta - \alpha)\) and \(1/\alpha\); call this circle \( \mathcal{C}_H(-1/(\beta - \alpha), 1/\alpha) \).

(3) Let \( k_G \) be the Nyquist gain of \(-G\). Then the exponential stability requirement on \((I - \beta^{-1}(G + \alpha I))^{-1}(G + \alpha I)\) can be stated as \( 0 \leq 1/(\beta - \alpha) < k_G \). A sufficient condition in terms of the \( \nu \)-index is given by
\[ \beta > \nu(-G) + \alpha. \tag{2.9} \]

(4) By manipulations similar to that for (2.8), a sufficient condition for \( \nu((I - \beta^{-1}(G + \alpha I))^{-1}(G + \alpha I)) \leq 0 \) can be shown as:
\[ a(\beta - \alpha) + (\beta - 2\alpha) \text{Re} \mu_{\min}(\hat{g}(j\omega)) - \| \hat{g}(j\omega) \|^2 \geq 0 \quad \text{for all } \omega \in \mathbb{R}. \tag{2.10} \]

This condition also has a circle interpretation. With \( \beta \geq \alpha \geq 0 \), (2.10) is equivalent to the generalized Nyquist diagram of \( G \), denoted by \( \mathcal{N}_G \), enclosed in a circle symmetric about the horizontal axis with intersections at \(-\alpha \) and \(\beta - \alpha \); call this circle \( \mathcal{C}_G(-\alpha, \beta - \alpha) \).

Given \( H \), a procedure based on the four conditions (2.7)–(2.10) above can be formulated for finding a stability condition on \( G \).

(1) Let the circle center at \((\rho, 0)\) with radius \( \eta \) be the circle of the least radius that strictly encloses \( \mathcal{N}_H \).

(2) Then the following condition
\[ \mathcal{N}_G \subset \begin{cases} \mathcal{C}_G \left( \frac{1}{\eta + \rho}, \frac{1}{\eta - \rho} \right) & \text{if } \eta > \rho, \\ \mathcal{C}_G \left( \frac{1}{2\eta}, \infty \right) & \text{if } \eta \leq \rho, \end{cases} \tag{2.11} \]

(\( \subset \) is a strict inclusion) implies that there exist \((\alpha, \beta), 0 \leq \alpha \leq \beta \), such that (2.7)–(2.10) are satisfied, which in turn implies the internal states of \( H \) and \( G \) converge to zero exponentially as \( t \rightarrow \infty \).

The last generalization that we shall need is that \( H \) and \( G \) in all the stability analysis above can be replaced by \((\xi^*/\xi)H \) and \((\xi/\xi^*)G \) where \( \xi \) is a polynomial with roots coincide with the unstable (non-minimum-phase) zeros of \( H \).
3. The control design problem

We now use the circle criterion presented in the last section for the design of stabilizing compensators. The approach is a generalization of [8,5] in which the small gain theorem was used. Assume the open loop plant can be described by (2.1), and can be decomposed into the sum (in the input/output sense) of an unstable finite dimensional portion and an exponentially stable infinite dimensional portion. As shown in [11], systems that satisfy the latter assumption are the ones that can be robustly stabilized, so no generality is lost.

The design procedure for the reduced order controllers by applying the circle stability criterion can be summarized in the following steps:

1. Decompose the infinite dimensional open loop plant $P_0$ into the sum of three subsystems:

$$P_0 = P_u + P_s + P_r$$

where $P_u$ is the unstable portion of the plant, $P_s$ is a finite dimensional approximation of the infinite dimensional stable portion of the plant and $P_r$ is the infinite dimensional residue that also contains any imprecise modeling in $P_u$ and $P_s$.

2. Define $P = P_u + P_s$. Denote the set of controllers that stabilize $P$ by $C(P)$. Let $K \in C(P)$. With the compensator, the transfer function around $P_r$ is $K(I - PK)^{-1}$.

3. Let $w$ denote the output signal from $K$. Then going around the loop, we have

$$w = K(I - PK)^{-1}P_tw = \frac{\xi^*}{\xi}K(I - PK)^{-1}\frac{\xi^*}{\xi}P_tw$$

where $\xi$ is either 1 or $d_u$, the characteristic polynomial of $P_u$. This system can be decomposed into a feedback interconnection with the forward system:

$$\hat{h} = \frac{\xi^*}{\xi}K(I - PK)^{-1}$$

and a feedback system:

$$\hat{g} = \frac{\xi^*}{\xi}P_r.$$  

Clearly, both systems are in $RH_m$. To remove the explicit dependence of $\hat{h}$ on $P_u$, a transformation from [8] is used which essentially subtracts $P_s$ from both $P$ and $K$ (so there is no net effect). Then

$$\hat{h} = \frac{\xi^*}{\xi}K_1(I - P_uK_1)^{-1}$$

where $K$ and $K_1$ are related by

$$K_1 = (I - KP_s)^{-1}K, \quad K = (I + K_1P_s)^{-1}K_1.$$  

(Without loss of generality, $P_u$ can be considered to be strictly proper.)

4. Choose $K$ such that the radius of the smallest circle that encloses $\mathcal{N}_H^\prime$ is minimized. A numerical procedure for computing this radius (and the corresponding $K$) will be presented below and the value only depends on $P_u$. Then the required level of approximation can be immediately obtained from (2.11) and (3.4).

The main computation in the above procedure is in finding $K$ in step (4). This can be posed as the following optimization problem:

$$\eta = \inf_{p} \| \hat{h} - pI \|_{H_m}.$$  

(3.7)
This problem is similar to the positive realness synthesis problem posed in [18], but the circle criterion provides a simpler graphical motivation.

First consider the $\xi = 1$ case. The complete set of stabilizing compensator for $P_u$ can be parameterized by

$$K_1 = (Y + M Q)(X + N Q)^{-1},$$

(3.8)

where $Q$ is any $H_\infty$ matrix and $M$, $N$, $X$, $Y$ and other relevant quantities are given by the following doubly coprime stable factorization of $P_u$, assuming $(A, B, C, 0)$ is a balanced (hence, minimal) realization of $P_u$ (for the details of this particular choice of doubly coprime factorization, see [8]):

$$\begin{align*}
P_u &= N M^{-1} = \tilde{M}^{-1} \tilde{N}, \\
\begin{bmatrix} M \\ N \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix} (sI - A + BF)^{-1} \begin{bmatrix} B \\ H \end{bmatrix}, \\
\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} F \\ -C \end{bmatrix} (sI - A + HC)^{-1} \begin{bmatrix} B \\ H \end{bmatrix},
\end{align*}$$

(3.9)

$$A \Sigma + \Sigma A^T = BB^T, \quad A^T \Sigma + \Sigma A = C^T C,$$

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0,$$

$$F = B^T \Sigma^{-1}, \quad H = \Sigma^{-1} C^T.$$

For simplicity, we have assumed that $P_u$ does not have poles on the j$\omega$-axis. When the assumption is false, a simple modification can be made but will not be done here. In general, this assumption is not an overly restrictive one since the j$\omega$-axis can always be shifted by a small amount so these poles are in the right half plane. With this assumption, the two Lyapunov equations in (3.9) are solvable [3]. Furthermore, $M$ and $\tilde{M}$ are inners, i.e.,

$$M^* M = \tilde{M}^* \tilde{M} = I.$$

After substituting the parametric form of the controller (3.8) into (3.7), the optimization problem becomes

$$\text{Find } Q \text{ to minimize } \eta = \| R - Q \|_{H_\infty}$$

(3.10)

where

$$R = \rho F_1 + F_2, \quad F_1 = M^* \tilde{M}^*, \quad F_2 = M^* Y.$$  

(3.11)

If $M$ and $\tilde{M}$ are not inners due to j$\omega$-axis poles in $P_u$, then to obtain the Nehari problem, the inner–outer factorization of $M$ and co-inner–outer factorization of $\tilde{M}$ need to be used [7, §7].

Note that $R$ is antistable, so this is the standard Nehari problem of finding the best stable approximation of an antistable $L_\infty$ matrix. By using (3.9), we can compute a state space representation for $F_1$ and $F_2$:

$$F_1 = \left( I - [B^T \quad H^T] \begin{bmatrix} sl - (A + BF)^T & F^T H^T \\ 0 & -(A + HC)^T \end{bmatrix} \right)^{-1} \begin{bmatrix} F^T \\ C^T \end{bmatrix},$$

(3.12a)

$$F_2 = -F (sl - A)^{-1} H.$$  

(3.12b)

It is easy to see that the controllability Gramian and the observability Gramian of $F_1$ are

$$\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & \Sigma \end{bmatrix}$$

and

$$\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^{-1} \end{bmatrix},$$

respectively, and, for $F_2$, both Gramians are $\Sigma^{-1}$. Hence,

$$\| \Gamma_{F_1} \| = 1, \quad \| \Gamma_{F_2} \| = 1/\sigma_-$$

(3.13)
where $\Gamma_F$ denotes the Hankel operator with symbol $F$ and $\sigma_\infty \triangleq \sigma_{\min}(P_u^*) = \min$ Hankel singular value of $P_u^*$ [8]. A bound for $\eta$ in (3.10) can now be stated:

$$\eta = \|\Gamma_R\| \leq \xi + 1/\sigma_\infty.$$  

(3.14)

Substitute the result back into (2.11); then $P_r$ is required to satisfy

$$\mathcal{N}_P \subseteq \mathcal{K}_P^\prime \left( -\frac{\sigma_\infty}{1 + \rho \sigma_\infty}, \sigma_\infty \right).$$

(3.15)

Since $\rho$ is arbitrary, the least conservative form of the above condition simply states that $\|P_r\|_{\infty} \leq \sigma_\infty$, which is the same as the small gain method in [5]. Hence, at least according to the bound (3.14), the circle criterion offers no advantage over the small gain approach. This is not too surprising since, intuitively, the circle criterion would be less conservative than the small gain criterion if the Nyquist plot is mostly in the right half plane (or the left half plane, if the signs in $H$ and $G$ are both reversed). But $H$ is non-minimum-phase due to the unstable open loop poles becoming the zeros, hence we expect $\nu(H)$ to be large (large excursion into the left half plane) and the advantage of the circle criterion is therefore lost.

Since the non-minimum-phase zeros impose fundamental limitation to the size of $\eta$, it may be advantageous to choose $\xi = d_u$ to change the unstable zeros in $\hat{h}$ to their stable mirror images. We have not proved this substitution would always result in less conservative bound in general, but the expectation is a reasonable one, and for a simple example to be discussed below, a better bound is obtained. In any case, this approach will do as well as the small gain approach since the $\rho = 0$ case coincides with the small gain method. We will first describe the general procedure of finding $\eta$ (for a given $\rho$) in this setting and then discuss the example.

With $\xi = d_u$, the same parameterization of $K_1$ can be used. The Nehari problem is again arrived at, except $F_1$ is now

$$F_1 = \frac{\xi}{\xi^*} M^* M^*.$$  

(3.16)

We are unable to derive an analytic bound in this case, but, numerically, $\eta$ can be computed for each $\rho$. The following algorithm can then be used to construct a stabilizing compensator.

**Algorithm 1.**

(1) Select a finite grid in $\mathbb{R}_+$ for $\rho$.

(2) For each $\rho$, find the corresponding $\eta = \|\Gamma_R\|$, with $R = \rho F_1 + F_2$, $F_1$ given by (3.16), and $F_2$ by (3.11).

(3) For the least conservative ($\rho, \eta$) pair (a possible measure is the diameter of the circle for the Nyquist plot for $(\xi/\xi^*)P_r: 1/(\eta - \rho) + 1/(\eta + \rho)$). Solve for $Q$ from the procedure in [7, pp. 127–128], and for $K_1$ from (3.8), and for $K$ from (3.6) (with $P_s$ selected so that $P_r$ is within the required bound).

For illustration of this algorithm, consider the following single-input/single-output system with a single unstable pole:

$$y = P_o(s)u = \left( \frac{f}{s - b} + P_c(s) + P_r(s) \right)u$$

(3.17)

where $P_o$ is finite dimensional and $P_r$ is infinite dimensional and both are stable. The examples in [8,4,5] are all of these form. Without loss of generality, assume $f > 0$. As stated before, $P_s$ will be incorporated in the control law and $P_r$ is considered as the residue. The antistable function $R$ in the Nehari problem is computed to be

$$R = \rho \left( 1 + 2b \frac{1 - 2b/\nu_p}{s - b} \right).$$
Therefore,

\[ \eta = \| \Gamma_R \| = \left| \frac{\rho - 2b}{f} \right|. \]

In particular, when \( \rho = b/f, \eta = \rho. \) In this case, from (2.11), it is sufficient to have

\[ \mathcal{N}_G \subset \mathcal{C}_G(-f/2b, \infty) \]

where \( G \triangleq (s-b)/(s+b)P_i; \) or, equivalently,

\[ \nu(G) < f/2b. \] (3.18)

Note that when the unstable mode is on the \( j\omega \)-axis, i.e., \( b = 0, \) the right hand bound tends to infinity. In this case, only the unstable mode needs to be used for the control design \( (P_i = 0); \) in fact, negative scalar output feedback can be used to stabilize the entire system. The right hand side of (3.18) is simply the Hankel singular value of the unstable portion, \( f/(s-b). \) In the small gain based method in [5], the stability condition is of the form:

\[ \| P_i(s) \|_{H_{\infty}} < f/2b. \] (3.19)

Since \( \nu((s-b)/(s+b)P_i(s)) \leq \| P_i(s) \|_{H_{\infty}}, \) (3.18) is a less stringent condition. In general, the method here will do no worse than the small gain method since it is included as a special case (with \( \rho = 0). \) However, whether it will always be less conservative is uncertain at this point.

4. Summary

This paper generalized the stabilizing compensator design method in [5] by using the circle criterion instead of the small gain criterion to analyze stability. Two methods are investigated. The first is shown to be identical to the small gain method, the second includes the small gain method as a special case, and, via an example, the stability condition for the unmodeled residue is shown to be less conservative.

Due to the Hilbert space generalization of the positive realness theorem, the input and output operators are required to be bounded. In [5], certain unbounded input/output operators are allowed. Extension in the current setting is under investigation by using the results in [17]. Another direction that merits investigation is in finding an approximation scheme for \( P_i \) given the circle that its Nyquist diagram must be enclosed in.

Bibliography