Necessary Conditions for Singularities in Three-Legged Platform-Type Parallel Mechanisms

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Abstract— Parallel mechanisms frequently contain an unstable type of singularity that has no counterpart in serial mechanisms. When the mechanism is at or near this type of singularity, it loses the ability to counteract external forces in certain directions. The determination of unstable singular configurations in parallel robots is challenging in general, and is usually tackled via exhaustive search of the workspace using an accurate analytical model of the mechanism kinematics.

This paper considers the singularity determination problem from a geometric perspective for planar and spatial three-legged parallel mechanisms. By using the constraints on the passive joint velocities, we derive a necessary condition for the unstable singularities. Using this condition, certain singularities can be found for certain type of platforms. As an example, new singular poses are discovered using this approach for a 6-DOF machining center.

Keywords—Parallel robot, parallel mechanism, unstable singularity, coordinate-free

I. INTRODUCTION

Parallel robots provide a stiff connection between the payload and the base structure, with pose accuracy that is superior to serial chain manipulators. The principal drawbacks concerning parallel robots are their limited workspace, and the complexity of singularity analysis [1], [2], [3]. In contrast to serial chain manipulators, singularities in parallel mechanisms have different manifestations. This issue has been studied in the multi-finger grasping context in [4], [5] and more recently for general parallel mechanisms in [6], [7], [8]. In [6], the singularities are separated into two broad classifications: end-effector and actuator singularities. The former is comparable to the serial arm case, where the end-effector loses a degree-of-freedom in the task space. The latter is defined when a certain task wrench cannot be resisted by active joint torques. Or equivalently, the task frame can move even when all the active joints are locked. These are called the unstable configurations in [8], [9] which correspond to unstable grasps in the multi-finger grasp literature. The unstable type of singularity is obviously unattractive, as unpredictable task motion could result.

The singularity condition in parallel mechanisms has been addressed using manipulability measures. For the multi-finger grasp, [4] presents a manipulability measure in the form of a Rayleigh Quotient. This work is later expanded to include passive finger joints in [7], [10]. In [11], a modified version of the fundamental grasping constraint is used to define three types of singularities. This is later used to develop a kinematic isotropy measure implemented as a design tool in [12]. In [5], the velocity-torque duality is used to find the composite Jacobian by post-multiplying the hand Jacobian by the pseudo-inverse of the grasp map. A pseudo-Riemannian metric is used to analyze manipulability in [6]. The method addresses passivity in the joint space as well as actuator redundancy.

Several authors have proposed methods to determine unstable configurations in platform manipulators through direct analysis of the forward kinematic constraints [13], [14], [15]. In [13], singularities of Griffith-Duffy type platform is investigated through the constraint equations from the forward kinematics. It was shown that the mechanism exhibits self-motion of the end-effector at every pose in its workspace.

Actuator redundancy is proposed to deal with singularities and improve mechanism isotropy in the development of a spherical parallel manipulator in [16]. Actuator redundancy is also used in [17], [18] to eliminate unstable singularities found after the construction of the Eclipse universal machining mechanism.

This paper considers the singularity determination problem for three-legged parallel mechanism from a coordinate-free perspective. By examining only the constraint matrix involving the passive joint velocities (kinematic constraints when all active joints are locked), we derive a geometric characterization the necessary condition for singularities for planar mechanisms and spatial mechanisms con-

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taining a passive spherical joint in every leg. The latter case includes the 6-DOF machining center developed by the Seoul National University [17]. Using the approach developed in this paper, we have found singular configurations hitherto unreported in the literature. Geometric characterization of singularities not only provides an more intuitive understanding of the singular configurations, but can also assist in the design of such mechanisms (e.g., move the singularity outside of the workspace) and planning for motion trajectories.

The rest of paper is organized as follows. Section II reviews the differential kinematics of parallel robots. Section III addresses the singularity of planar parallel mechanisms. Section IV considers the singularity of three-legged spatial algorithms with a passive spherical joint in each leg.

II. DIFFERENTIAL KINEMATICS OF PARALLEL ROBOTS

This section considers the differential kinematics of platform type of mechanisms. Consider a platform with k legs. Let the joint angles (both active and passive) of each leg be denoted by the vector \( \theta_i \). Denote the spatial velocity of the platform by \( v_T \) and the Jacobian for leg \( i \) by \( J_i \) (from \( \theta_i \) to \( v_T \)), we have the following differential kinematic relationship:

\[
\begin{bmatrix}
J_1(\theta_1) & 0 & \ldots & 0 \\
0 & J_2(\theta_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_k(\theta_k)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\vdots \\
\dot{\theta}_k
\end{bmatrix}
=
\begin{bmatrix}
I \\
\vdots \\
I
\end{bmatrix}
\begin{bmatrix}
v_T \\
\vdots \\
v_T
\end{bmatrix}
\]

(1)

Note that \( \Lambda \) is always full (column) rank. The kinematic relationship can be equivalently written as

\[
v_T = \Lambda^T \dot{\theta}_a \quad (2)
\]

\[
0 = \Lambda^T \dot{\theta}_p \quad (3)
\]

where \( \Lambda^T \) is the pseudo-inverse of \( \Lambda \) and \( \Lambda \) is the annihilator of \( \Lambda \). Now, we collect all the active joint velocities together in \( \dot{\theta}_a \) and the passive joint velocities together in \( \dot{\theta}_p \), and then partition \( J_C \) and \( J_T \) accordingly:

\[
J_C = \begin{bmatrix} J_{C_a} & J_{C_p} \end{bmatrix} \quad J_T = \begin{bmatrix} J_{T_a} & J_{T_p} \end{bmatrix} \quad (4)
\]

Assume that \( J_{C_p} \) is square (number of passive joints is equal to the number of kinematic constraints); otherwise, there may be internal self-motion. If \( J_{C_p} \) is full rank, then Eq. (3) can be used to solve for \( \dot{\theta}_p \):

\[
\dot{\theta}_p = -J^{-1}_{C_p} J_{C_a} \dot{\theta}_a. \quad (5)
\]

Substituting into (3), we have

\[
v_T = (J_{T_a} - J_{T_p} J^{-1}_{C_p} J_{C_a}) \dot{\theta}_a \quad (6)
\]

where \( J_T \) is the composite manipulability Jacobian.

We can now classify the singularities as follows:

1. Unstable Singularity: This corresponds to configurations at which \( J_{C_p} \) becomes singular (maximum singular value of \( J_T \) is infinite).
2. Unmanipulable Singularity: This corresponds to configurations at which \( J_T \) loses rank (minimum singular value of \( J_T \) is zero).

It is convenient to visualize \( J_T \) as mapping a unit ball in the active joint velocity space to an ellipsoid in the spatial task velocity space. At an unmanipulable singularity, the ellipsoid becomes degenerate (the length of one or more axes become zero, implying that the ellipsoid has zero volume). At an unstable singularity, the ellipsoid becomes infinite (the length of one or more axes become infinite, implying that arbitrary task velocity is possible even when active joints are locked).

The unstable singularity, unique to parallel mechanisms, presents a dangerous situation. When the mechanism moves through these poses, it is unable to resist specific task wrenches, which can result in undesirable and unavoidable end-effector motions. For platform types of parallel manipulators, it is common to find six-leg type of platforms (e.g., Gough-Steward platform) and three-leg type of platforms (e.g., Eclipse machining center developed by the Seoul National University). Six-legged platforms tend to have small work volume but less singularity problem. Three-legged platforms can have much larger work volume but singularity becomes a serious consideration. For example, as described in [17], singularities within the workspace were found after the Eclipse machining center has been designed and constructed. In this paper, we focus on the structure of \( J_{C_p} \) for three-legged mechanisms and develop necessary geometric conditions under which \( J_{C_p} \) becomes singular.

III. PLANEAR MECHANISMS

Consider a three-legged planar mechanism with all active joints locked. The free joints for each leg may be either both revolute (RR) or one revolute and one prismatic (in either order, RP or PR), but not both prismatic [19] (see Figure 1 where only the passive joints are shown).

For the RR case, the differential kinematics from the free joint angular velocities to the end effector
spatial velocity is:
\[
\begin{align*}
\dot{\theta}_E &= \dot{\theta}_1 + \dot{\theta}_4 = \dot{\theta}_2 + \dot{\theta}_5 = \dot{\theta}_3 + \dot{\theta}_6 \quad (7)
\frac{d\vec{p}_{OE}}{dt} &= Z \times \vec{p}_{1E} \dot{\theta}_1 + Z \times \vec{p}_{AE} \dot{\theta}_4 \\
&= Z \times \vec{p}_{2E} \dot{\theta}_2 + Z \times \vec{p}_{5E} \dot{\theta}_5 \\
&= Z \times \vec{p}_{3E} \dot{\theta}_3 + Z \times \vec{p}_{6E} \dot{\theta}_6 \quad (8)
\end{align*}
\]
where \( \frac{d\vec{p}_{OE}}{dt} \) is the vectorial derivative of \( \vec{p}_{OE} \) in the inertial frame [20] and \( Z \) is the unit vector out of the page. Using (7), we can write (8) as
\[
\begin{align*}
\frac{d\vec{p}_{OE}}{dt} &= Z \times \vec{p}_{1E} \dot{\theta}_1 + Z \times \vec{p}_{5E} \dot{\theta}_5 \\
&= Z \times \vec{p}_{2E} \dot{\theta}_2 + Z \times \vec{p}_{5E} \dot{\theta}_5
\end{align*}
\]
This set of equalities can be written as constraints on the passive joint velocities:
\[
\begin{align*}
Z \times \vec{p}_{14} \dot{\theta}_1 - Z \times \vec{p}_{25} \dot{\theta}_2 + Z \times \vec{p}_{45} &= 0 \\
Z \times \vec{p}_{14} \dot{\theta}_1 - Z \times \vec{p}_{36} \dot{\theta}_3 + Z \times \vec{p}_{46} &= 0.
\end{align*}
\]
We can then form the passive constraint Jacobian, \( J_{C_P} \):
\[
J_{C_P} \dot{\theta}_P = \begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 \\
\dot{\theta}_4
\end{bmatrix}
\]
For both PR and RP cases, the passive constraint Jacobian is
\[
J_{C_P} \dot{\theta}_P = \begin{bmatrix}
\vec{h}_1 & -\vec{h}_2 & 0 & \vec{Z} \times \vec{p}_{45} \\
\vec{h}_1 & 0 & -\vec{h}_3 & \vec{Z} \times \vec{p}_{46}
\end{bmatrix}
\begin{bmatrix}
\dot{d}_1 \\
\dot{d}_2 \\
\dot{d}_3 \\
\dot{\theta}_E
\end{bmatrix}
\]
where \( \vec{d}_i \) is the translational velocity of the \( i \)th prismatic joint and \( \vec{h}_i \) denotes the unit vector along the joint.

In all cases, the passive constraint Jacobian is of the form
\[
J_{C_P} = \begin{bmatrix}
a & b & 0 & d_1 \\
a & 0 & c & d_2
\end{bmatrix}
\]
where \( a, b, c, d_i \)'s are vectors lying in the same plane. The mechanism is at a singular pose if and only if the Jacobian is rank deficient.

The following theorem provides a geometric characterization of the conditions under which \( J_{C_P} \) in (12) loses rank.

**Theorem 1**: Consider \( J_{C_P} \) in (12). \( J_{C_P} \) is rank deficient (\( \text{rank}(J_{C_P}) < 4 \)) if and only if the following condition holds:
The line segments \( \alpha a, d_1 - \beta b, d_2 - \gamma c \), where \( \alpha, \beta, \gamma \) are arbitrary real numbers, intersect at least one point, including infinity (see Figure 2).

**Proof**: The matrix \( J_{C_P} \) loses rank if and only if either
case 1: The submatrix \( J_{C_P} = \begin{bmatrix} a & b & 0 & d_1 \\ a & 0 & c & d_2 \end{bmatrix} \) loses rank, or,
case 2: \( J_{C_P} \) is full rank and \( \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \) is in the range of \( J_{C_P} \).

Case 1 is true if and only if \( a, b, c \) are collinear.
In this case, the line segments \( \alpha a, d_1 - \beta b, d_2 - \gamma c \)
are parallel, which can be considered as intersection at infinity.

Case 2 is true if and only if there exists a solution \((\alpha, \beta, \gamma)\) for the following equation:

\[
\begin{bmatrix}
  a & b & 0 \\
  a & 0 & c
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma
\end{bmatrix} =
\begin{bmatrix}
  d_1 \\
  d_2
\end{bmatrix}
\]  

(13)

which is the condition for the three line segments intersecting at a common point. 

In general, the singularity condition is easy to understand intuitively: the torque at the point of intersection cannot be resisted, or, equivalently, there can be a rotation about the point of intersection.

Note that the above geometric characterization only represents a necessary condition for singularities since the singular configurations may not be kinematically feasible, i.e., there may not be any active joint configurations such that these conditions are satisfied. For example, for an RPR (underline denotes the active joint) mechanism, the three line segments are never parallel except for a specific choice of parameters.

For the RR case, the line intersection condition is

\[
\alpha \hat{e} \times \vec{p}_{14} - \beta \hat{e} \times \vec{p}_{25} = \hat{e} \times \vec{p}_{45} \quad (14)
\]

\[
\alpha \hat{e} \times \vec{p}_{14} - \gamma \hat{e} \times \vec{p}_{56} = \hat{e} \times \vec{p}_{46} \quad (15)
\]

Since the mechanism is planar, these equations can be equivalently written as

\[
\alpha \vec{p}_{14} = \vec{p}_{45} + \beta \vec{p}_{25} \quad (16)
\]

\[
\alpha \vec{p}_{14} = \vec{p}_{46} + \gamma \vec{p}_{56} \quad (17)
\]

which is simply the intersection condition of the three legs. Figure 3 shows various singular configurations in this case. Again, not all the configurations may be attainable for a given mechanism.

Note that cases (b), (c), (d) are simply special cases of the general line intersecting case (e) where two of the line segments are collinear.

For the RP and PR cases, let \(h_1^+, h_2^+\), and \(h_3^+\) denote the planar unit vectors orthogonal to \(h_1, h_2,\) and \(h_3\), respectively. The line intersection condition then reduces to

\[
\alpha h_1^+ = \vec{p}_{45} + h_2^+ \quad (18)
\]

\[
\alpha h_1^+ = \vec{p}_{46} + h_3^+ \quad (19)
\]

which is the intersection condition for the line parallel to \(h_1^+\) passing through point 4, the line parallel to \(h_2^+\) passing through point 5, and the line parallel to \(h_3^+\) passing through point 6. Examples for RP and PR cases are shown in Figure 4.

### IV. Spatial Mechanisms

Consider a three-legged spatial mechanism with all active joints locked. Assume each leg contains one spherical free joint and one revolute or prismatic joint (RS, PS, SR, or SP) (see Figure 5 for RS and PS types, SR and SP are similar).

For the RS and SR cases, the differential kinematics from the free joint angular velocities to the end effector spatial velocity is:

\[
\vec{\omega}_{\text{AE}} = \dot{\vec{h}}_1 + \vec{\omega}_{A1} = \dot{\vec{h}}_2 + \vec{\omega}_{A2} = \dot{\vec{h}}_3 + \vec{\omega}_{A3} + (\mathbf{X})
\]
\[ J_{CP_v} \hat{\theta}_p = \begin{bmatrix} \hat{h}_1 \times \hat{p}_{14} & -\hat{h}_2 \times \hat{p}_{25} & 0 & \hat{p}_{54} \times \hat{p}_{64} \\ \hat{h}_1 \times \hat{p}_{14} & 0 & -\hat{h}_3 \times \hat{p}_{36} & \hat{p}_{64} \times \hat{h}_n \end{bmatrix} \begin{bmatrix} \dot{\hat{h}}_1 \\ \dot{\hat{h}}_2 \\ \dot{\hat{h}}_3 \\ \dot{\hat{h}}_n \end{bmatrix} \] (23)

Let \( \hat{h}_n \) be the unit vector perpendicular to both \( \hat{p}_{54} \) and \( \hat{p}_{64} \). We can simplify \( J_{CP_v} \) by performing an invertible transformation (assuming \( \hat{p}_{54} \) and \( \hat{p}_{64} \) are not collinear, otherwise, \( J_{CP_v} \) is always singular):

\[ J_{CP_v} = J_{CP_v} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{p}_{54} \times \hat{p}_{64} & \hat{h}_n \end{bmatrix} \] (24)

After elementary operations, \( J_{CP_v} \) can be written as

\[ J_{CP_v} = \begin{bmatrix} \hat{h}_1 \times \hat{p}_{14} & \hat{h}_2 \times \hat{p}_{25} & 0 & \hat{p}_{54} \times \hat{h}_n & \hat{h}_n \end{bmatrix} \begin{bmatrix} \dot{\hat{h}}_1 \\ \dot{\hat{h}}_2 \\ \hat{h}_1 \times \hat{p}_{14} & 0 & \hat{h}_3 \times \hat{p}_{36} & \hat{p}_{64} \times \hat{h}_n & \hat{h}_n \end{bmatrix} \] (25)

For the PS and SP cases, the differential kinematics becomes

\[ \frac{d\hat{p}_{56}}{dt} = \hat{h}_1 \hat{p}_{14} - \hat{p}_{54} \times \hat{w}_{56} \]

\[ \frac{d\hat{p}_{65}}{dt} = \hat{h}_2 \hat{p}_{25} - \hat{p}_{65} \times \hat{w}_{65} \]

\[ \frac{d\hat{p}_{64}}{dt} = \hat{h}_3 \hat{p}_{36} - \hat{p}_{64} \times \hat{w}_{64} \] (26)

It follows that the passive constraint Jacobian is:

\[ J_{CP_v} \hat{\theta}_p = \begin{bmatrix} \hat{h}_1 \times \hat{p}_{14} & -\hat{h}_2 \times \hat{p}_{25} & 0 & \hat{p}_{54} \times \hat{p}_{64} \times \hat{w}_{54} \times \hat{h}_n \end{bmatrix} \begin{bmatrix} \dot{\hat{h}}_1 \\ \dot{\hat{h}}_2 \\ \hat{h}_1 \times \hat{p}_{14} & -\hat{h}_3 \times \hat{p}_{36} & \hat{p}_{64} \times \hat{h}_n \end{bmatrix} \] (27)

After applying the transformation as in (24) and performing elementary column operations, we obtain

\[ J_{CP_v} = \begin{bmatrix} \hat{h}_1 \hat{p}_{14} & 0 & \hat{p}_{54} \times \hat{h}_n & \hat{h}_n \end{bmatrix} \begin{bmatrix} \dot{\hat{h}}_1 \\ \hat{h}_1 \times \hat{p}_{14} & 0 & \hat{h}_3 \times \hat{p}_{36} & \hat{p}_{64} \times \hat{h}_n & \hat{h}_n \end{bmatrix} \] (28)

In both SR and SP cases, the passive constraint Jacobian is of the form

\[ J_{CP_v} = \begin{bmatrix} a & b & 0 & c & d_1 & h \\ a & c & d_2 & 0 & h \end{bmatrix} \] (29)

where \( a, b, c, d_1, d_2, h \) are Cartesian vectors, \( d_1 \) and \( d_2 \) are not collinear, \( d_1 \) and \( d_2 \) are both orthogonal to \( h \). The mechanism is at a singular pose if and only if the Jacobian is rank deficient.
The following theorem extends Theorem 1 to the spatial case, providing a geometric condition under which \( J_{C_{R_i}} \) in (29) loses rank.

**Theorem 2:** Consider \( J_{C_{R_i}} \) in (29). Let \( a, b, c \) be decomposed as

\[
\begin{align*}
a &= a_1 + a_2 \\
b &= b_1 + b_2 \\
c &= c_1 + c_2
\end{align*}
\]

where \( a_1, b_1, c_1 \) are collinear with \( h \) and \( a_2, b_2, c_2 \) are orthogonal to \( h \).

Then \( J_{C_{R_i}} \) is rank deficient (\( \text{rank}(J_{C_{R_i}}) < 4 \)) if and only if the following condition holds:

The line segments \( \alpha a_2, d_1 - \beta b_2, d_2 - \gamma c_2 \), where \( \alpha, \beta, \gamma \) are arbitrary real numbers, intersect at least one point, including infinity (see Figure 6).

![Fig. 6. Singularity Condition for Spatial Mechanism](image)

**Proof:** The matrix \( J_{C_{R_i}} \) can be further column-reduced to

\[
J_{C_{R_i}} = \begin{bmatrix} a_2 & b_2 & 0 & d_1 & h & 0 \\ a_2 & 0 & c_2 & d_2 & 0 & h \end{bmatrix}
\]

This matrix loses rank if and only if either case 1: The submatrix

\[
\begin{bmatrix} a_2 & b_2 & 0 \\ a_2 & 0 & c_2 \end{bmatrix}
\]

loses rank, or,

- case 2: \( J_{C_{R_i}} \) is full rank and \( \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \) is in the range of \( J_{C_{R_i}} \).

Case 1 is true if and only if \( a_2, b_2, c_2 \) are collinear. In this case, the line segments \( \alpha a_2, d_1 - \beta b_2, d_2 - \gamma c_2 \) are parallel, which can be considered as intersection at infinity.

Case 2 is true if and only if there exists a solution \((\alpha, \beta, \gamma)\) for the following equation:

\[
\begin{bmatrix} a_2 & b_2 & 0 \\ a_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\]

which is the condition for the three line segments intersecting at a common point. \( \Box \)

There are various special cases that aid visualizing the singularity condition. Denote by \( L_a \) the line segment parallel to \( a \) and passing through the intersection of \( d_1 \) and \( d_2 \). \( L_b \) the line segment parallel to \( b \) passing through the tip of \( d_1 \), and \( L_c \) the line segment parallel to \( c \) passing through the tip of \( d_2 \). Consider the following special cases of singularity conditions:

1. One of the lines is trivial (i.e., \( a_2 = 0, b_2 = 0, \) or \( c_2 = 0 \)). Then the other two lines always intersect as shown in Figure 7.
2. Two of the lines overlap, as shown in Figure 8.
3. The three lines intersect at a vertex of the triangle formed by \( d_1, d_2, \) and \( d_1 - d_2 \), as shown in Figure 9.
4. The three lines are parallel, as shown in Figure 10.

For RS mechanisms, examples of these special cases are shown in Figure 11. In the case that \( h_1, h_2, h_3, \) and \( h_n \) are all parallel. Then we recover exactly the planar situation. In Case 1, the rotational axis and the leg attached to it are in the plane of the platform. In Case 2, two of the rotational axes are orthogonal to the platform and the legs are collinear as shown. In Case 3, two of the rotational axes are again orthogonal to the platform but the legs intersect at the third spherical joint. In Case 4, all three rotational axes are orthogonal to the platform and the legs are parallel. For PS mechanisms, if \( h_1, h_2, h_3 \) are coplanar with the platform, we have the same singular configuration as in the PR case in Figure 4.

An example of the RS mechanism is the 5-face machining center developed at the Seoul National University called Eclipse [17], shown in Figures 12 and 13. The three posts can be driven around the track (at points \( a, b, \) and \( c \) and joints 1-3 can translate up and down the posts. The passive joints are the revolute joints at points 1-3 on the posts, and the three spherical joints at 4-6. Let \( h_i, i = 1, \ldots, 3, \) be the vectors tangential to the circular track, located at each vertical post, \( p_{14}, p_{25}, p_{36} \) be the vectors of the arms supporting the platform, \( p_{45}, p_{45} \), \( p_{46} \) be the vectors linking the spherical joints on the platform, and \( h_i \) be the unit vector normal to the platform.

Define the planes formed by each support arm and the corresponding pivot axis as:

\[
\begin{align*}
\pi_1 &= \text{the plane containing } h_1 \text{ and } p_{14} \\
\pi_2 &= \text{the plane containing } h_2 \text{ and } p_{25} \\
\pi_3 &= \text{the plane containing } h_3 \text{ and } p_{36}
\end{align*}
\]
The vectors in Theorem 2 are then given by

\[ a = h_1 \times p_{14} \text{ (normal of } \pi_1) \]
\[ a_2 = a - (a \cdot h_n)h_n \]
(a projected onto the platform plane)
\[ b = h_2 \times p_{25} \text{ (normal of } \pi_2) \]
\[ b_2 = b - (b \cdot h_n)h_n \]
(b projected onto the platform plane)
\[ c = h_3 \times p_{36} \text{ (normal of } \pi_3) \]
\[ c_2 = c - (c \cdot h_n)h_n \]
(c projected onto the platform plane)

\[ d_1 = p_{54} \times h_n \]
(vector on the platform plane normal to \( p_{54} \))
\[ d_2 = p_{64} \times h_n \]
(vector on the platform plane normal to \( p_{64} \)).

To characterize the singularities, we first define the following geometric conditions:

(A1) \(: p_{45} \in \pi_1 \)
(A2) \(: p_{45} \in \pi_2 \)
(B1) \(: p_{64} \in \pi_1 \)
(B2) \(: p_{64} \in \pi_3 \)
(C1) \(: p_{56} \in \pi_3 \)
(C2) \(: p_{56} \in \pi_2 \).

The singularities are given by the four cases above:

1. One of the lines, \( L_a, L_b, L_c \), is trivial (i.e., a point). This case corresponds to \( a, b, \) or \( c \) parallel to \( h_n \). Geometrically, this means that \( \pi_1, \pi_2, \) or \( \pi_3 \) (planes formed by the support arms and the post revolute axes) coplanar with the platform plane. This is the only singularity that has been previously identified. Note that the singularity condition only involves one of the supporting arms. This case corresponds to

(A1,B1), (A2,C2), or (B2,C1).

2. Two of the lines, \( L_a, L_b, L_c \), overlap. This case corresponds to the line connecting two spherical joints on the platform lying in the intersection of the corresponding planes formed by the support arms and the post revolute axes. Note that the singularity involves two of the supporting arms and the spherical joints connected to the arms. The geometric characterization is given by:

(A1,A2), (B1,B2), or (C1,C2).

3. The lines, \( L_a, L_b, L_c \), intersect at a spherical joint on the platform. The geometric condition involves two of the supporting arms but all three spherical joints:

\( (A2,B2), (A1,C1), \) or \( (B1,C2) \).

4. The lines, \( L_a, L_b, L_c \), are parallel. This corresponds to the normals of \( \pi_1, \pi_2, \pi_3 \), projected onto the platform plane being parallel. In this case, all three supporting arms and all three spherical joints are involved.

Fig. 7. Case 1: One of the Lines is Trivial
Fig. 8. Case 2: Two of the Lines Overlap

Fig. 9. Case 3: Three Lines Intersect at a Vertex

Fig. 10. Case 4: Three Lines are Parallel

Fig. 11. Examples of Singularities in RS Mechanisms
Fig. 12. Picture of Eclipse (used with permission from the Seoul National University)

Fig. 13. Schematics of Eclipse
V. CONCLUSIONS

Parallel mechanism offers advantages such as superior load to weight ratio and stiffness. However, finding and avoiding unstable configurations in the workspace is in general a difficult task. Singularity determination is usually done through an exhaustive search of the workspace. This procedure is time consuming, may miss some singularities due to the granularity of the search, and does not offer ready geometric insight of these configurations. In this paper, we present a coordinate-free approach to finding the necessary conditions for singularities for certain three-legged platform type parallel mechanisms. Our current research addresses using the geometric characterization of singularities to assist parallel manipulator design.

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