Since $\beta = \{1 ; 1\}$ is effective w.r.t. $K$ and
\[
\det A(-1)_{(1, 1)} = \det \begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix} \neq 0
\]
s $= -1$ is not a fixed mode.

In comparison to other generalized criteria of fixed modes (e.g., [6], [7]), the criterion in this note is more convenient in some cases. In Tarokh's criterion, polynomial algebraic operation must be performed before a state pole is checked. In Hu's criterion, much higher dimensions occur in the matrix calculations because a state-space model, instead of MFD, is adopted.

REFERENCES


Time Domain and Frequency Domain Conditions for Strict Positive Realness

JOHN T. WEN

Abstract—This note states various time and frequency domain conditions pertaining to strictly positive real systems and establishes their relationship.

I. INTRODUCTION

The concept of positive realness has appeared in many branches of system analysis. Positive real systems, also called passive systems, are systems that do not generate energy. One of the most useful properties of positive real systems is the equivalence between the time domain characterization in terms of the Lyapunov equations and the frequency domain condition of the nonnegativity of the Hermitian part of the transfer function. This fact is popularly known as the positive realness lemma. The application of this powerful result ranges from network analysis, adaptive control, nonlinear control, and robustness analysis. When a system dissipates energy, it is sometimes called strictly positive real. In contrast to positive realness, the relationship between the time and frequency domain conditions in this case is less well understood. In [1], this issue is resolved for single-input/single-output (SISO) systems. The goal of this note is to fill the rest of the void by providing a comprehensive list of time domain and frequency domain conditions relating to multivariate strictly positive real systems and demonstrating the connection between them. A similar but independent work in this direction has also appeared in [2]. The condition based on the Lyapunov equation is proposed to be the new definition for strict positive realness. Then the main theorem in this note can be considered as the positive realness lemma for strictly positive real systems.

II. MAIN RESULT

The system under consideration is a linear time-invariant (LTI) system, denoted by $\Sigma$, with the minimal (controllable and observable) state domain (state-space) representation
\[
x(t) = Ax(t) + Bu(t), \quad x(0) = x_0
\]
\[
y(t) = Cx(t) + Du(t)
\]
and the frequency domain (transfer function) representation
\[
y(s) = (D + C(sI - A)^{-1}B)u(s)
\]
where $x \in R^n$, $u, y \in R^m$. The quadruplet $(A, B, C, D)$ is an internal parameter set of $\Sigma$. In the subsequent analysis, $\sigma_{\text{min}}(B) > 0$ is assumed. This assumption is equivalent to either: 1) Ker $B = \{0\}$, or 2) $m < n$ and $B$ is of full rank. If this condition is not satisfied, then the input contains redundancy which can always be eliminated without loss of generality by taking a linear combination of the inputs and using it as the new control vector. Since the transfer function is required to be square, a linear combination of the output is also needed.

The usual notations of $\geq 0$ and $> 0$ are used to denote positive semidefiniteness and positive definiteness of matrices, respectively. The symbols $\mu_{\text{min}}(A)$ and $\sigma_{\text{max}}(A)$ are defined as the minimum eigenvalue of symmetrized $A$ (i.e., $1/2(A + A^T)$) and minimum matrix singular value, respectively. The term coercivity is used synonymously with bounded invertibility in the space under consideration.

The main result below relates various possible conditions for strict positive realness of $\Sigma$.

**Theorem 1:** Given an exponentially stable LTI system $\Sigma$ with an internal parameter set $(A, B, C, D)$ and transfer function $T(s)$. Assume $\sigma_{\text{min}}(B) > 0$. Consider the following statements.

1) There exist $P > 0, P \in R^{m \times m}, \mu_{\text{min}}(L) \geq \epsilon > 0, Q \in R^{m \times m}, W \in R^{m \times m}$ that satisfy the Lyapunov equations
\[A^T P + PA = -Q^T Q - L \quad (2.3a)
\]
\[B^T P - C = W^T Q \quad (2.3b)
\]
\[W^T W = D + D^T \quad (2.3c)
\]
1') Same as 1) except $L$ is related to $P$ by
\[
L = 2\mu P
\]
for some $\mu > 0$.

2) There exists $\eta > 0$ such that for all $\omega \in R$
\[T(j \omega) + T^*(j \omega) > \eta I. \quad (2.5)
\]

3) For all $\omega \in R$
\[T(j \omega) + T^*(j \omega) > 0. \quad (2.6)
\]
4) For all $\omega \in R$
\[T(j \omega) + T^*(j \omega) > 0. \quad (2.7a)
\]
and
\[
\lim_{\omega \to \infty} \omega^2(T(j \omega) + T^*(j \omega)) > 0. \quad (2.7b)
\]
5) The system $\Sigma$ can be realized as the driving point impedance of a multiport dissipative network.

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6) The Lur'e equations with \( L = 0 \) are satisfied by the internal parameter set \((A + \mu I, B, C, D)\) corresponding to \( T(j\omega - \mu) \) for some \( \mu > 0 \).

7) For all \( \omega \in R \), there exists \( \mu > 0 \) such that

\[
T(j\omega - \mu) + T(j\omega - \mu) \geq 0.
\]  

(2.8)

8) There exist a positive constant \( \rho \) and a constant \( \xi(x_0) \in R, \xi(0) = 0 \), such that for all \( T \geq 0 \)

\[
\int_0^T u^T(t)y(t) dt \geq \xi(x_0) + \rho \int_0^T \|u(t)\|^2 dt.
\]  

(2.9)

9) There exists a positive constant \( \gamma \) and a constant \( \xi(x_0), \xi(0) = 0 \), such that for all \( T \geq 0 \)

\[
\int_0^T e^{\gamma t}u^T(t)y(t) dt \geq \xi(x_0).
\]  

(2.10)

10) There exists a positive constant \( \alpha \) such that the following kernel is positive in \( L_2(R_+; R^{m \times m}) \)

\[
K(t-s) = D\delta(t-s) + Ce^{(\alpha + \alpha H)(t-s)}Bi(t-s).
\]  

(2.11)

where \( \delta \) and \( I \) denote the Dirac delta function and the step function, respectively.

11) The following kernel is coercive in \( L_2(0, T; R^{m \times m}) \), for all \( T \).

\[
K(t-s) = D\delta(t-s) + Ce^{(\alpha + \alpha H)(t-s)}Bi(t-s).
\]  

(2.12)

These statements are related as follows:

\[
\text{(1)} \quad \Rightarrow \quad \text{(2)} \quad \Rightarrow \quad \text{(8)} \quad \Rightarrow \quad \text{(11)}
\]

(2) \( \Rightarrow \) (11)

Consider the optimization problem of finding \( u \in L_2(-\infty, \infty; R^m) \) to minimize

\[
J_u = \int_{-\infty}^{\infty} \left( -2 \delta^T(j\omega)F^*Fx(j\omega) + 2\hat{\delta}^T(j\omega)\tilde{F}(j\omega) \right) d\omega
\]

where the superscript \( * \) denotes complex conjugate transposition and \( \hat{x}, \tilde{y}, \hat{u} \), and \( \tilde{u} \) are the Fourier transforms of \( x, y, u, \) and \( u \), respectively. By writing \( \hat{x} \) in terms of the initial condition, the optimization index can be expanded as

\[
J_u = \int_{-\infty}^{\infty} \left\{ -(j\omega I - A)^{-1}x_0 + (j\omega I - A)^{-1}B\hat{u}(j\omega) + B\hat{(j\omega I - A)^{-1}}B^*F((j\omega I - A)^{-1}x_0
\]

\[
+ ((j\omega I - A)^{-1}B\hat{u}(j\omega) + \hat{\delta}^T(j\omega)\{(C(j\omega I - A)^{-1})B + D\}^T
\]

\[
+ (C(j\omega I - A)^{-1}B + D\}^T\hat{\delta}(j\omega)
\]

\[
+ 2\hat{\delta}^T(j\omega)C(j\omega I - A)^{-1}x_0 \right\} d\omega
\]

The first equation implies \( P > 0 \). By identifying \( L \) with \( F^*F \) and choosing \( F^*F > 0 \) and

\[
\delta^T(j\omega)F^*(j\omega) < \frac{\eta}{\|((j\omega I - A)^{-1}B\|^2_{\text{norm}}}
\]

condition (1) is proved. 

(1) \( \Rightarrow \) (2)

(When \( D > 0 \))

Given the Lur'e equations, compute the Hermitian part of the transfer function as follows:

\[
T(j\omega) + \hat{T}(j\omega) = D + D^T + C(j\omega I - A)^{-1}B + B^*(j\omega I - A)^{-1}C^T
\]

\[
= W^*W + B^*P - W^*Q(j\omega I - A)^{-1}B
\]

\[
+ B^*(j\omega I - A)^{-1}(PB - QW)
\]

\[
= W^*W + B^*(j\omega I - A)^{-1}((j\omega I - A)^{-1}P
\]

\[
+ P(j\omega I - A))(j\omega I - A)^{-1}B
\]

\[
- W^*Q(j\omega I - A)^{-1}B - B^*(j\omega I - A)^{-1}Q^*W
\]

\[
= W^*W + B^*(j\omega I - A)^{-1}((Q^* + L(j\omega I - A))B
\]

\[
- W^*Q(j\omega I - A)^{-1}B - B^*(j\omega I - A)^{-1}Q^*W
\]

\[
= (W^* - B^*(j\omega I - A)^{-1}Q^*)W - Q(j\omega I - A)^{-1}B
\]

\[
+ B^*(j\omega I - A)^{-1}L(j\omega I - A)^{-1}B \geq 0.
\]
Assume condition (2) is false. Then there exist \( \{u_k\} \) \( \|u_k\| = 1 \), and \( \{\omega_k\} \) such that

\[
0 \leq \langle T(j\omega_k) + T^*(j\omega_k) \rangle u_k, u_k \leq \frac{1}{n}.
\]

As \( n \to \infty \), if \( \omega_k \to \infty \), then

\[
\langle T(j\omega_k) + T^*(j\omega_k) \rangle u_k, u_k \to (D_{u_k}, u_k) \geq \mu_{\infty}(D) > 0
\]

which is a contradiction since the left-hand side converges to zero. Hence, \( u_k \) and \( \omega_k \) are both bounded sequences and therefore contain convergent subsequences \( u_{k_0} \) and \( \omega_{k_0} \). Let the limits be \( u_0 \) and \( \omega_0 \). Then

\[
\langle T(j\omega_0) + T^*(j\omega_0) \rangle u_0, u_0 = 0.
\]

This implies

\[
W_{u_0} Q(j\omega_0 - A)^{-1} Bu_0 = 0
\]

which implies

\[
L^{1/2}(j\omega_0 - A)^{-1} Bu_0 = 0.
\]

Since \( L > 0 \), the second equality implies

\[
(j\omega_0 - A)^{-1} Bu_0 = 0.
\]

Substituting back to the first equality yields

\[
W_{u_0} = 0.
\]

The positive definiteness of \( W \) (by the assumption \( D > 0 \)) implies contradiction. Hence, condition (2) is satisfied.

(2) \( \Rightarrow \) (8)

Since (2) \( \Rightarrow \) (1), the Lur'\'e equation holds. Let

\[
V(x) = \frac{1}{2} x^TPx.
\]

Then

\[
V(x(t)) = x(t)^TPAx(t) + x(t)^TPBu(t)
\]

\[
= -\frac{1}{2} x(t)^TX(t) - \frac{1}{2} \|Qx(t)\|^2
\]

\[
+ u(t)^TPx(t) + u(t)^TPWQx(t)
\]

\[
= -\frac{1}{2} x(t)^TX(t) - \frac{1}{2} \|Qx(t)\|^2
\]

\[
- u(t)^TDu(t) + u(t)^TPWQx(t) + u(t)^TPy(t)
\]

\[
\leq -\frac{1}{2} \|x(t)\|^2 + u(t)^TPy(t) - \frac{1}{2} \|Qx(t) - WUu(t)\|^2
\]

\[
\leq -\frac{1}{2} \|x(t)\|^2 + u(t)^TPy(t).
\]

By integrating both sides [3] for all \( T \geq 0 \)

\[
\int_0^T u(t)^TPy(t) dt \geq -V(x_0).
\]

(13.21)

Since (2.5) remains valid if \( D \) is replaced by \( D - \epsilon \) for \( \epsilon \) sufficiently small, (12.13) holds with \( y \) replaced by

\[
y(t) = Cx + (D - \epsilon)u(t).
\]

Then (2.13) becomes

\[
\int_0^T u(t)^TPy(t) dt \geq \int_0^T \|u(t)\|^2 dt - V(x_0).
\]

Identifying \( -V(x_0) \) with \( \xi(x_0) \) and \( \epsilon \) with \( \rho \) in (2.9), condition (8) follows.

(8) \( \Rightarrow \) (2)

Let \( T \to \infty \) in (2.9), then

\[
\int_0^T u(t)^TPy(t) dt \geq \xi(x_0) + \rho \int_0^\infty \|u(t)\|^2 dt.
\]

In particular, for \( x_0 = 0 \),

\[
\int_0^\infty u(t)^TPy(t) dt \geq \rho \int_0^\infty \|u(t)\|^2 dt.
\]

By Plancherel's theorem,

\[
\int_0^\infty \|u(t)\|^2 dt = \int_0^\infty \|u(t)\|^2 dt
\]

for all \( \theta \in L^2 \). Suppose that for each \( \eta > 0 \), there exists \( \omega \in C \) and \( \omega_0 \in R \) such that

\[
\omega^T T(j\omega_0) w < \eta \|w\|^2.
\]

By the continuity of \( \omega^T T(j\omega_0) w \) in \( \omega \), there exists an interval \( \Omega \) around \( \omega_0 \) of length \( r \) such that

\[
\omega^T T(j\omega_0) w < \eta \|w\|^2
\]

for all \( \omega \in \Omega \). Let

\[
\theta(j\omega_0) = \begin{cases} \omega & \text{if } \omega \in \Omega \\ 0 & \text{otherwise} \end{cases}
\]

Clearly, \( \theta \in L^2 \). Then

\[
\int_0^\infty \|\theta(j\omega)\|^2 d\omega = \int_0^\infty \|\theta(j\omega)\|^2 d\omega < \eta \|w\|^2.
\]

and

\[
\rho \int_0^\infty \|\theta(j\omega)\|^2 d\omega = \rho \|w\|^2.
\]

If \( \eta < \rho \), this is a contradiction. Hence, there exists \( \eta > 0 \) such that (2.5) holds.

(8) \( \Rightarrow \) (11)

Condition (11) follows directly from condition (8).

(11) \( \Rightarrow \) (8)

The implication is obvious if \( x_0 = 0 \). In the proof of (8) \( \Rightarrow \) (2), \( x_0 \) is taken to be zero. Therefore, for \( x = 0 \), (11) \( \Rightarrow \) (2). It has already been shown that (2) \( \Rightarrow \) (8). Hence, (11) \( \Rightarrow \) (2) \( \Rightarrow \) (8).

(11') \( \Rightarrow \) (11)

By definition.

(11) \( \Rightarrow \) (11')

(If \( D = 0 \))

If \( D = 0 \), then \( W = 0 \). Rewrite (2.3a) as

\[
A^TP + PA = -Q^TQ - 2\mu P - 2\mu P.
\]

For \( \mu \) small enough,

\[
Q^TQ - L - 2\mu P \geq 0.
\]

Hence, there exists \( Q_t \) such that

\[
A^TP + PA = -Q_t^TQ_t - 2\mu P.
\]

Since (2.3b) is independent of \( Q_t \) when \( D = 0 \), (11') is proved.

(11') \( \Rightarrow \) (6)

By straightforward manipulation.

(6) \( \Rightarrow \) (7)

Same as in (1) \( \Rightarrow \) (2) except \( L \) is replaced by \( 2\mu P \).

(7) \( \Rightarrow \) (6)

Standard positive realness lemma (see [8]).

(4) \( \Rightarrow \) (7)

For \( \mu > 0 \) sufficiently small, \( A - \mu I \) remains strictly stable. Now, by direct substitution

\[
T(j\omega - \mu) + T^*(j\omega - \mu)
\]

\[
= D + D^T + C(j\omega - A - \mu I)^{-1} B + B^T(-j\omega - A - \mu I)^{-1} C^T
\]

\[
= T(j\omega) + T^*(j\omega) + \mu[C(j\omega - A)^{-1} B + B(-j\omega - A - \mu I)^{-1} C^T] - Q(j\omega - A)^{-1} Q^T C^T.
\]
Therefore, for any $w \in C^n$,

$$2w^*(T(j\omega - \mu)w) \geq 2w^*T(j\omega)w - 2\mu \|C\| \|B\| \|w\|^2.$$

Since

$$\|\mu \| [\|A\| - \|A\|] [x]\| \|w\|^2.$$

It follows [7]

$$\|\|A\| [x]\| \|w\|^2.$$

Then

$$\|\|A\| [x]\| \|w\|^2.$$

By (2.7a), for all $\omega \in \Omega$, $\Omega$ is compact in $R$, there exists $k > 0$, $k$ dependent on $\Omega$, such that

$$2w^*T(j\omega)w \geq k\|w\|^2.$$

By (2.7b), for $\omega$ sufficiently large, there exists $g > 0$ such that

$$2w^*T(j\omega)w \geq \frac{g}{\omega^2} \|w\|^2.$$

Hence, there exists $\omega_1 \in R$ large enough so that (2.14) and (2.15) hold with some $g$ and $k$ dependent on $\omega_1$. Then, for $\|A\| \leq \omega_1$,

$$2w^*T(j\omega - \mu)w \geq k\|w\|^2 - 2\mu \|C\| \|B\| \|w\|^2 - \mu \|\|A\| - \|A\|\| [x]\| \|w\|^2.$$

(2.16)

and for $\omega > \omega_1$

$$2w^*T(j\omega - \mu)w \geq \frac{g}{\omega^2} \|w\|^2 - 2\mu \|C\| \|B\| \|w\|^2 - \mu \|\|A\| - \|A\|\| [x]\| \|w\|^2.$$

(2.17)

The terms in curly brackets in (2.16) and (2.17) are finite. Hence, there exists $\mu$ small enough such that (2.16) and (2.17) are both nonnegative, proving condition (7).

(7) $\Rightarrow$ (4)

From (7) $\Rightarrow$ (6), the minimal realization $(A, B, C, D)$ associated with $T(j\omega)$ satisfies the Lur'e equation with $L = 2wP$. Following the same derivation as in (1) $\Rightarrow$ (2), for all $w \in C^*$, we have

$$w^*(T(j\omega + T^*(j\omega))w)$$

$$= w^*(T^*(j\omega)w + T^*(j\omega - \mu))w$$

$$+ 2w^*B^*(j\omega - \mu)P(j\omega - \mu) - \mu Bw$$

$$\geq 2w^*B^*(j\omega - \mu)P(j\omega - \mu) - \mu Bw$$

$$\geq 2\mu \|B\| \|A\| \|w\|^2.$$

Since $P$ is positive definite and, by assumption, $\alpha_m(B) > 0$, $T(j\omega)$ is positive for all $\omega \in R$.

It remains to show (2.7b). Multiply both sides of the inequality above by $\omega^2$, then

$$\omega^2w^*(T(j\omega)w + T^*(j\omega))w \geq 2\mu \|B\| \|A\| \|w\|^2.$$

As $\omega^2 \rightarrow \infty$, the lower bound converges to $2\mu \|B\| \|A\| \|w\|^2$, which is positive.

(7) $\Rightarrow$ (5)

If (2.8) is satisfied, $T(j\omega - \mu)$ corresponds to the driving point impedance of a multiport passive network [8]. Hence, $T(j\omega)$ corresponds to the impedance of the same network with all $C$ replaced by $C$ in parallel with a resistor of conductance $\mu C$ and $L$ replaced by $L$ in series with a resistor of resistance $\mu L$. Since all $L$, $C$ elements are now lossy, or dissipative, $T(j\omega)$ is the driving point impedance of a dissipative network.

(5) $\Rightarrow$ (7)

Reversing the argument, if $T(j\omega)$ is the driving point impedance of a dissipative network, all $L$ and $C$ elements are lossy. Hence, by removing sufficiently small series resistance in $L$ and parallel conductance in $C$, the network would remain passive. Hence, again by [8], condition (7) is satisfied.

(6) $\Rightarrow$ (9)

Let

$$V(t, x) = \frac{1}{2} e^{\mu x t} P x.$$

Then

$$V(t, x(t))$$

$$\leq \frac{1}{2} \|P\| e^{\mu x t} P x(t) + \frac{1}{2} e^{\mu x t} P (PA + A^T P)x(t) + e^{\mu x t} P y(t).$$

Choose $0 < \gamma < 2\|P\|$. Then by comparison principle, for all $T \geq 0$,

$$\int_0^T e^{\mu u(t)} y(t) dt \geq - \int_0^T x^T P x_{12},$$

(9) $\Rightarrow$ (6)

Define

$$u(t) = e^{\mu x t} y(t)$$

$$x(t) = e^{\mu x t} x(t).$$

(2.18)

The corresponding transfer function is

$$T(t, j\omega)$$

$$= D + C (j\omega - \mu) - \frac{1}{2} I$$

$$= T_{11} (j\omega).$$

By setting $T = \infty$ and $x_{12} = 0$ in (2.10),

$$\int_0^\infty u(t) x(t) dt \geq 0.$$

By Plancherel's theorem,

$$\int_0^\infty \dot{u}(t) x(t) dt \geq 0.$$
Since this holds true for all \( \hat{u}(j\omega) \in L_2 \),
\[
T_1(j\omega) + T_2^*(j\omega) \geq 0.
\]
Equivalently,
\[
T\left(j\omega - \frac{\gamma}{2}\right) + T^*\left(j\omega - \frac{\gamma}{2}\right) \geq 0
\]
proving (7).

(9) \iff (10)
Use the transformation in (2.18), then condition (10) follows directly from condition (9) with \( \alpha = \gamma/2 \).

(10) \iff (9)
If \( x_0 = 0 \), (10) \iff (9) is obvious. Since in the proof of (9) \iff (6), only the \( x_0 = 0 \) case is considered, it follows, for the \( x_0 = 0 \) case, (10) \iff (9) \iff (6). It has already been shown that (6) \iff (9). Hence, (10) \iff (6) \iff (9).

(2) \iff (4) \iff (3)

The implications are obvious.

Which condition in Theorem 1 should be used as a definition for strict positive reaility? At present, there appears to be no consensus in the literature. Condition (3) has been used as a definition for strict positive reality [9], [6]. As seen in Theorem 1, it is in general too weak to imply the Lur'e equations which are usually needed for stability analysis. In [1], condition (5) has been used as the definition for strict positive reality. For SISO systems, condition (4) has been noted to be necessary [10] and later necessary and sufficient [1] for condition (5). Therefore, if a frequency domain definition of strict positive reality is sought, condition (4) appears more appropriate. Condition (7) has been used by [10] as a definition for strict positive reality for both SISO and MIMO systems. Condition (8) was termed \( u \)-strictly-passive for nonlinear systems in [11]. For the \( D > 0 \) case, (2) \iff (1) has also been shown for distributed parameter systems [12], [13]. Because of the importance of the Lur’e equations that are used for the eventual stability analysis, we propose to use condition (1) as the definition of strict positive reality.

III. SUMMARY

Various characterizations for strict positive real systems have been examined in this note. Their mutual relationship has been clarified via a three-tier diagram; each tier represents a set of equivalent conditions successively stronger than the lower tier.

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REFERENCES