will be such that \( \delta J = 0 \) for arbitrary small variations of \( x_k(t_o) \) around this value. For this to be the case, we choose
\[
\lambda_k(t_o) = 0,
\]  
(2.4.1)
which simply says that the influence of small changes in \( x_k(t_o) \) on \( J \) is zero. Again we have simply traded one boundary condition, \( x_k(t_o) \) given, for another, (2.4.1). Boundary conditions like (2.4.1) are sometimes called "natural boundary conditions."

However, the necessary condition (2.3.13), \( \partial H / \partial u = 0 \), needs additional justification for the problem with terminal constraints. In Section 2.3, we derived it under the assumption that \( \delta u(t) \), \( t_o < t < t_f \) is arbitrary. In the present case, \( \delta u(t) \) is not completely arbitrary; the set of admissible \( \delta u(t) \) is restricted by the constraints
\[
\delta x_i(t_f) = 0, \quad i = 1, \ldots, q,
\]  
(2.4.2)
where we define "admissible" \( \delta u(t) \), generally, as those \( \delta u(t) \) which satisfy all constraints of the problem, for example, (2.4.2).

Now, it is still possible to determine influence functions for the performance index exactly as in Section 2.3. In this section we shall designate these influence functions with a superscript "f." However, since \( x_i(t_f) \) for \( i = 1, \ldots, q \) are specified, it is consistent to regard
\[
\phi = \phi[x_{q+1}, \ldots, x_n]_{t=t_f}.
\]  
(2.4.3)
Thus (cf., Equations (2.3.7) through (2.3.9), we have (for \( \delta x(t_o) = 0 \))
\[
\delta J = \int_{t_o}^{t_f} \left[ \partial L / \partial u + (\lambda^{(u)})^T \partial f / \partial u \right] \delta u(t) \, dt,
\]  
(2.4.4)
where
\[
\lambda^{(u)} = -\left( \frac{\partial f}{\partial x} \right)^T \lambda^{(u)} - \left( \frac{\partial L}{\partial x} \right)^T
\]  
(2.4.5)
\[
\lambda^{(u)}(t_f) = \begin{cases} 
0 ; & j = 1, \ldots, q \\
\frac{\partial \phi}{\partial x_j} \bigg|_{t=t_f} ; & j = q + 1, \ldots, n.
\end{cases}
\]  
(2.4.6)
Suppose that, instead of \( J = \phi[x(t_f)] + \int_{t_o}^{t_f} L(x,u,t) \, dt \), the performance index was \( J = x_i(t_f) \); i.e., the \( i \)th component of the state vector at the final time. We could then determine influence functions for \( x_i(t_f) \) by specializing the relations above; we would put \( \phi = x_i(t_f) \) and \( L(x,u,t) = 0 \). We shall designate these influence functions with a superscript "i." Analogous to Equations (2.4.4), (2.4.5), and (2.4.6), we have
\[ \delta x_i(t_f) = \int_{t_0}^{t_f} (\lambda^{(i)})^T \frac{\partial f}{\partial u} \delta u(t) \, dt, \quad (2.4.7) \]

where

\[ \lambda^{(i)} = - \left( \frac{\partial f}{\partial x} \right)^T \lambda^{(i)}, \quad (2.4.8) \]

\[ \lambda_j^{(i)}(t_f) = \begin{cases} 0; & i \neq j, \\ 1; & i = j, \quad j = 1, \ldots, n \end{cases} \quad (2.4.9) \]

We could, in fact, determine \( q \) sets of such influence functions for \( i = 1, \ldots, q \) (see Appendix A4).

We shall now construct a \( \delta u(t) \) history that decreases \( J \), i.e., produces \( \delta J < 0 \), and satisfies the \( q \) terminal constraints (2.4.2). Multiply each of the \( q \) equations in (2.4.7) by an undetermined constant, \( \nu_i \), and add the resulting equations to (2.4.4):

\[ \delta J + \nu_i \delta x_i(t_f) = \int_{t_0}^{t_f} \left\{ \frac{\partial L}{\partial u} + [\lambda^{(j)} + \nu_i \lambda^{(i)}]^T \frac{\partial f}{\partial u} \right\} \delta u \, dt. \quad (2.4.10) \]

Now choose

\[ \delta u = -k \left\{ \left( \frac{\partial f}{\partial u} \right)^T [\lambda^{(j)} + \nu_i \lambda^{(i)}] + \left( \frac{\partial L}{\partial u} \right)^T \right\}, \quad (2.4.11) \]

where \( k \) is a positive scalar constant, and substitute this expression into (2.4.10), as follows:

\[ \delta J + \nu_i \delta x_i(t_f) = -k \int_{t_0}^{t_f} \left\{ \left( \frac{\partial f}{\partial u} \right)^T (\lambda^{(j)} + \nu_i \lambda^{(i)}) + \left( \frac{\partial L}{\partial u} \right)^T \right\}^2 \, dt < 0, \quad (2.4.12) \]

which is negative unless the integrand vanishes over the whole integration interval.

Next, we determine the \( \nu_i \)'s so as to satisfy the terminal constraints (2.4.2). Substituting (2.4.11) into (2.4.7), we have

\[ 0 = \delta x_i(t_f) = -k \int_{t_0}^{t_f} \left[ (\lambda^{(i)})^T \frac{\partial f}{\partial u} \left( \lambda^{(j)} + \nu_j \lambda^{(j)} \right) + \left( \frac{\partial L}{\partial u} \right)^T \right] \, dt \]

\[ = \int_{t_0}^{t_f} \left[ (\lambda^{(i)})^T \frac{\partial f}{\partial u} \lambda^{(j)} + \left( \frac{\partial L}{\partial u} \right)^T \right] \, dt + \nu_j \int_{t_0}^{t_f} \left[ (\lambda^{(i)})^T \frac{\partial f}{\partial u} \right] \, dt, \]

\[ \nu_i \delta x_i = \sum_{i=1}^{q} \nu_i \delta x_i. \]

\[ \Repeated indices indicate summation over the range of that index, for example: \]