\[ J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} [L(x,u,t) + \lambda^T(t)f(x,u,t) - \lambda^T \dot{x}] \, dt. \]  

The differential of (2.7.1), taking into account differential changes in the terminal time, \( t_f \), is

\[
dJ = \left( \frac{\partial \phi}{\partial t} \, dt_f + \frac{\partial \phi}{\partial x} \, dx \right)_{t=t_f} + (L)_{t=t_f} \, dt_f + \int_{t_0}^{t_f} \left[ \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \delta x \right. \\
\left. + \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) \delta u - \lambda^T \delta \dot{x} \right] \, dt. \tag{2.7.2}
\]

Integrating (2.7.2) by parts and collecting terms gives

\[
dJ = \left[ \left( \frac{\partial \phi}{\partial t} + L \right) dt_f + \frac{\partial \phi}{\partial x} \, dx \right]_{t=t_f} - \left[ \lambda^T \delta x \right]_{t=t_f} + \left[ \lambda^T \delta x \right]_{t=t_0} \\
\int_{t_0}^{t_f} \left[ \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \delta x + \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) \delta u \right] \, dt. \tag{2.7.3}
\]

Now \( \delta x \), the variation in \( x \), means "for time held fixed," so \( dx \), the differential in \( x \), may be written (see Figure 2.7.1)

\[ dx(t_f) = \delta x(t_f) + \dot{x}(t_f) \, dt_f. \tag{2.7.4} \]

\[ 
\begin{align*}
\text{Nominal path} \\
\text{Neighboring path}
\end{align*}
\]

**Figure 2.7.1.** Relationship between \( dx(t_f) \), \( \delta x(t_f) \), and \( dt_f \).

From (2.7.4), we have \( \delta x(t_f) = dx(t_f) - \dot{x}(t_f) \, dt_f \); substituting this into (2.7.3) and collecting terms gives

\[
dJ = \left[ \left( \frac{\partial \phi}{\partial t} + L + \lambda^T \dot{x} \right) dt_f + \left( \frac{\partial \phi}{\partial x} - \lambda^T \right) dx \right]_{t=t_f} + \lambda^T(t_0) \delta x(t_0) \\
\int_{t_0}^{t_f} \left[ \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \delta x + \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) \delta u \right] \, dt. \tag{2.7.5}
\]
Now, as in Section 2.4, we consider that
\[ x_i(t_f) ; \quad i = 1, \ldots, q \text{ are specified}, \] (2.7.6)
and, hence, it is consistent to consider \( \phi \) to be a function only of the unspecified state variables:
\[ \phi = \phi[x_j(t_f), t_f] ; \quad j = q + 1, \ldots, n. \] (2.7.7)

Next, we choose the functions \( \lambda(t) = \lambda^{(i)}(t) \) to make the coefficients of \( \delta x(t) \), and \( dx(t_f) \) vanish in (2.7.5)
\[ \dot{\lambda}^{(i)} = - \left( \frac{\partial L}{\partial x} \right)^\top - \left( \frac{\partial f}{\partial x} \right)^\top \lambda^{(i)}, \] (2.7.8)
\[ \lambda_j^{(i)}(t_f) = \begin{cases} 0, & j = 1, \ldots, q, \\ \left( \frac{\partial \phi}{\partial x_j} \right)_{t=t_f}, & j = q + 1, \ldots, n. \end{cases} \] (2.7.9)

This choice of \( \lambda(t) \) leaves (2.7.5) in the form
\[ dJ = \left( \frac{\partial \phi}{\partial t} + L + f^\top \lambda^{(i)} \right)_{t=t_f} dt_f + \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial u} + \left[ \lambda^{(i)} \right]^\top \frac{\partial f}{\partial u} \right] \delta u dt, \] (2.7.10)
where we have placed \( \delta x(t_o) = 0 \) since \( x(t_o) \) is given.

Now, as in Section 2.4, let us consider the change in \( x_i(t_f) \), \( i = 1, \ldots, q \) for arbitrary \( \delta u(t) \). Using the concept of influence (adjoint) functions (see Appendix A3) we have
\[ dx_i(t_f) = [f_i]_{t=t_f} dt_f + \int_{t_0}^{t_f} \left[ \lambda^{(i)}(t) \right]^\top \frac{\partial f}{\partial u} \delta u dt, \] (2.7.11)
where
\[ \dot{\lambda}^{(i)} = - \left( \frac{\partial f}{\partial x} \right)^\top \lambda^{(i)}, \] (2.7.12)
\[ \lambda_j^{(i)}(t_f) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \] (2.7.13)

Note that Equation (2.7.11) may be regarded as a special case of (2.7.10) by replacing \( \phi \) with \( x_i \) and placing \( L = 0 \).

We will now construct a \( \delta u(t) \) history and select a value for \( dt_f \) that produces \( dJ < 0 \), and satisfies \( dx_i(t_f) = 0, i = 1, \ldots, q \). Multiply each of the \( q \) equations (2.7.11) by an undetermined constant, \( \nu_i \), and add the resulting equations to (2.7.10)
\[ dJ + \nu_i dx_i(t_f) = \left\{ \frac{\partial \phi}{\partial t} + L + \left[ \lambda^{(i)} \right]^\top f + \nu_i f_i \right\}_{t=t_f} dt_f + \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial u} + \left( \lambda^{(i)} + \nu_i \lambda^{(i)} \right)^\top \frac{\partial f}{\partial u} \right] \delta u dt. \] (2.7.14)