Problem 1. Consider

$$\bar{J} = \Phi(x(t_f), t_f) + \int_0^{t_f} L(x, u, t) \, dt$$

and $t_f$ as a control parameter. What is the variation of $J$ due to a variation of $t_f$ when all optimality conditions in Section 2.4 are to be satisfied? From this, derive the condition

$$\frac{\partial \Phi}{\partial t_f} = -H(t_f)$$

directly. [HINT:

$$d\bar{J} = \frac{\partial \Phi}{\partial x} \frac{dx}{dt} dt_f + \frac{\partial \Phi}{\partial t_f} dt_f + L dt_f.]$$

MINIMUM-TIME SOLUTIONS. In many problems, the performance index of interest is the elapsed time to transfer the system from its initial state to a specified state. In this case, we may place

$$\phi = 0, \quad L = 1, \quad (2.7.28)$$

which implies that

$$J = t_f - t_o. \quad (2.7.29)$$

The minimum-time control program is obtained, then, by solving the two-point boundary-value problem:

$$\begin{align*}
\dot{x} &= f(x, u, t); \quad x(t_o) \text{ given} \ (n \text{ initial conditions}), \quad (2.7.30) \\
\dot{\lambda} &= -(f_x)^T \lambda; \quad x_j(t_f) \text{ specified}; \quad j = 1, \ldots, q; \quad (2.7.31) \\
\lambda_j(t_f) &= 0, j = q + 1, \ldots, n \ (n \text{ terminal conditions}), \quad (2.7.32) \\
0 &= f^T_u \lambda \ (m \text{ optimality conditions}), \quad (2.7.33) \\
(\lambda^T f)_{t=t_f} &= -1.
\end{align*}$$

Note that there are $2n$ boundary conditions for the $2n$ differential equations (2.7.30)–(2.7.31), $m$ optimality conditions (2.7.32) for the $m$ control variables, $u$, and one transversality condition (2.7.33) for the terminal time, $t_f$. The unspecified values of $\lambda_j(t_f), j = 1, \ldots, q$, which we have called $\nu_j$ above, are part of the solution.

Note, also, that at least one state variable must be specified at $t = t_o$ and at $t = t_f$ or the minimum-time problem makes no sense.

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\footnote{If $x_j(t_o)$ is not specified, we have $\lambda_j(t_o) = 0$.}
Example 1. Minimum-time paths through a region of position-dependent vector velocity (Zermelo’s problem).† A ship must travel through a region of strong currents. The magnitude and direction of the currents are known as functions of position:

\[ u = u(x,y) \quad \text{and} \quad v = v(x,y), \]

where \((x,y)\) are rectangular coordinates and \((u,v)\) are the velocity components of the current in the \(x\) and \(y\) directions, respectively. The magnitude of the ship’s velocity relative to the water is \(V\), a constant. The problem is to steer the ship in such a way as to minimize the time necessary to go from a point \(A\) to a point \(B\).

The equations of motion are

\[ \dot{x} = V \cos \theta + u(x,y), \]
\[ \dot{y} = V \sin \theta + v(x,y), \]

where \(\theta\) is the heading angle of the ship’s axis relative to the (fixed) coordinate axes, and \((x,y)\) represents the position of the ship.

The Hamiltonian of the system is

\[ H = \lambda_x (V \cos \theta + u) + \lambda_y (V \sin \theta + v) + 1, \]

(2.7.36)

So the Euler-Lagrange equations are

\[ \dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_x \frac{\partial u}{\partial x} - \lambda_y \frac{\partial v}{\partial x}, \]

(2.7.37)

\[ \dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x \frac{\partial u}{\partial y} - \lambda_y \frac{\partial v}{\partial y}, \]

(2.7.38)

\[ 0 = \frac{\partial H}{\partial \theta} = V(-\lambda_x \sin \theta + \lambda_y \cos \theta), \quad \Rightarrow \tan \theta = \frac{\lambda_y}{\lambda_x}. \]

(2.7.39)

Since the Hamiltonian (2.7.36) is not an explicit function of time, \(H = \text{constant}\) is an integral of the system. Furthermore, since we are minimizing time, this constant must be 0. We may solve (2.7.36) and (2.7.39) for \(\lambda_x\) and \(\lambda_y\):

\[ \lambda_x = \frac{-\cos \theta}{V + u \cos \theta + v \sin \theta}, \]

(2.7.40)

\[ \lambda_y = \frac{-\sin \theta}{V + u \cos \theta + v \sin \theta}. \]

(2.7.41)

†For another derivation, using vector notation, see Section 3.2, Example 2, which treats the problem in three dimensions (e.g., for an aircraft in a region of strong winds).