\[ \dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial f}{\partial x} - \frac{\partial L}{\partial x}, \quad (2.8.9) \]

\[ \lambda^T(t_f) = \left( \frac{\partial \Phi}{\partial x} \right)_{t=t_f} = \left( \frac{\partial \Phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x} \right)_{t=t_f}, \quad (2.8.10) \]

\[ \left( \frac{\partial \Phi}{\partial t} + L + \lambda^T \dot{x} \right)_{t=t_f} = \left( \frac{d\Phi}{dt} + L \right)_{t=t_f} = 0, \quad (2.8.11) \]

where

\[ \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} \dot{x}. \]

As a result of this choice of \( \lambda(t) \), (2.8.8) is simplified to

\[ dJ = \int_{t_0}^{t_f} \frac{\partial H}{\partial u} \delta u \, dt + \lambda^T(t_o) \delta x(t_o) - H(t_o) \, dt_o. \quad (2.8.12) \]

Clearly, as before, \( \lambda^T(t_o) \) is the influence vector on \( J \) of changes in initial conditions \( \delta x(t_o) \), while \( \partial H/\partial u \) is a set of impulse-response functions indicating how \( J \) would change as a result of unit impulses in the controls at any point in the interval \( t_o \leq t \leq t_f \).

For a stationary value of \( J \), clearly, we have

\[ \frac{\partial H}{\partial u} = \lambda^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u} = 0, \quad t_o \leq t \leq t_f, \quad (2.8.13) \]

and if a component \( x_k(t_o) \) is not specified, we have \( \lambda_k(t_o) = 0 \).

For minimum time, \( t_f - t_o \), we may let \( \phi[x(t_f) t_f] = 0 \) and \( L = 1 \), so that condition (2.8.11) becomes

\[ \left( \frac{d\Phi}{dt} + 1 \right)_{t=t_f} = 0. \quad (2.8.14) \]

As in Section 2.6, the \( q \) constants \( \nu \) must be determined to satisfy the terminal constraints (2.8.2). The condition (2.8.14) is the extra condition needed to determine the final time \( t_f \).

**In summary**, a set of necessary conditions for \( J \) to have a stationary value is

\[ \dot{x} = f(x,u,t) \quad (2.8.15) \]

\[ \dot{\lambda} = -\left( \frac{\partial H}{\partial x} \right)^T = -\left( \frac{\partial f}{\partial x} \right)^T \lambda - \left( \frac{\partial L}{\partial x} \right)^T \quad (2.8.16) \]

\[ \text{†An argument regarding admissibility, similar to the one made in Section 2.7, must be made to justify (2.8.13).} \]
\[ 0 = \left( \frac{\partial H}{\partial u} \right)^T = \left( \frac{\partial f}{\partial u} \right)^T \lambda + \left( \frac{\partial L}{\partial u} \right)^T \quad (2.8.17) \]

\( x_k(t_o) \) given, or \( \lambda_k(t_o) = 0 \) \quad (2.8.18)

\[ \lambda(t_f) = \left( \frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x} \right)^T_{t=t_f} \quad (2.8.19) \]

\[ \Omega = \left[ \frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left( \frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x} \right)^T f + L \right]_{t=t_f} = 0 \quad (2.8.20) \]

\[ \psi[x(t_f)t_f] = 0 \quad (2.8.21) \]

The optimality condition (2.8.17) determines the \( m \)-vector \( u(t) \). The solution to the \( 2n \) differential equations (2.8.15) and (2.8.16) and the choice of the \( q + 1 \) parameters \( \nu \) and \( t_f \) are determined by the \( 2n + 1 + q \) boundary conditions (2.8.18)–(2.8.21). Needless to say, this boundary-value problem is, in general, not very easy to solve.

Notice, however, that if we were to specify \( \nu \) instead of \( \psi \), and \( t_f \) instead of \( \Omega \), (2.8.18) and (2.8.19) provide \( 2n \) boundary conditions for a fixed-terminal-time, two-point boundary-value problem of order \( 2n \). By changing values of \( \nu \) and \( t_f \), it may be possible to bring \( \psi \) and \( \Omega \) to zero at \( t = t_f \) (see Chapter 7, Section 3).